

Structure of chambers cut out by Veronese arrangements of hyperplanes in the real projective spaces

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Introduction

Consider several lines in the real projective plane. The lines divide the plane in various chambers. We are interested in the arrangement of the chambers. This problem is quite naive and simple. But if the cardinality m of the lines are large, there are no way to control in general. If we assume that no three lines meet at a point (line arrangements **in general position**), the situation does not improve much. In this paper we always assume this. Let us observe when m is small. Since we are mathematicians, we start from $m = 0$: The projective plane itself. When $m = 1$, what remains is the euclidean plane, of course. When $m = 2$, the plane is divided in two di-angular chambers.

When $m = 3$, the lines cut out four triangles; if we think one of the lines is the line at infinity, then the remaining two can be considered as two axes dividing the euclidean plane in four parts. Another explanation: if we think that the three lines bound a triangle, then around the central triangle there is a triangle along each edge/vertex.

When $m = 4$, four triangles and three quadrilaterals are arranged as follows; if we think one of the lines is the line at infinity, then the remaining three lines bound a triangle, and around this triangle there are a quadrilateral along each edge, and a triangle kissing at each vertex. Another explanation: put the four lines as the symbol $\#$, then around the central quadrilateral are a triangle along each edge, and a quadrilateral kissing at each pair of opposite vertices. The situation is already not so trivial.

When $m = 5$, there is a unique pentagon, and if we see the arrangement centered at this pentagon, the situation can be described simply: there are five triangles adjacent to five edges, and five quadrilaterals kissing at five vertices. If we think one of the lines is the line at infinity, the remaining four lines can not be like $\#$ since the five lines should be in general position; the picture does not look simple, but one can find a pentagon. Thanks to the central pentagon, we can understand well the arrangement.

When $m = 6$, there are four types of arrangements. The simplest one has a hexagon surrounded by six triangles along the six edges; there are six quadrilaterals kissing at the six vertices, and three quadrilaterals away from the hexagon. Another one with icosahedral symmetry: If we identify the antipodal points of the dodeca-icosahedron projected from the center onto a sphere we get six lines in the projective plane: there are $20/2 = 10$ triangles and $12/2 = 6$ pentagons. We do not describe here the two other types.

For general m there are so many types, and we can not find a way to control them, unless the m lines bound an m -gon, from which we can see the arrangement. We can think this m -gon as the *center* of this arrangement, and we would like to study higher dimensional versions of this kind of arrangements with center, if such a chamber exists. Before going further, since we are mathematicians, we start from the very beginning: the (-1) -dimensional projective space is an empty set; well it is a bit difficult to find something interesting. The 0-dimensional projective space is a point; this point would be the center, OK. Now we proceed to the projective line. m points on the line divide the line into m intervals. (In this case ‘in general position’ means ‘distinct’.) Very easy, but there is no *center* in the arrangement!

How about 3-dimensional case? We have m planes in general position (no four planes meet at a point). $m = 0$: The projective space itself. When $m = 1$, what remains is the euclidean space, of course. When $m = 2$, the space is divided into two di-hedral chambers. When $m = 3$, the space is divided into four tri-hedral chambers.

When $m = 4$, the planes cut out eight tetrahedra; if we think one of the planes is the plane at infinity, then the remaining three can be considered as the coordinate planes dividing the euclidean space in eight parts. Another explanation: if we think that the four planes bound a tetrahedron, then around the central tetrahedron there is a tetrahedron adjacent to each face (kisses at the opposite vertex), and a tetrahedron touching along a pair of opposite edges: $1 + 4 + 6/2 = 8$.

When $m = 5$, if we think one of the planes is the plane at infinity, then the remaining four bounds a tetrahedron. Around this central tetrahedron, there is a (triangular) prism adjacent to the face, a prism touching along an edge, and a tetrahedron kissing at a vertex. ($1T + 4P + 6P + 4T$.) Since the central tetrahedron is bounded only by four planes, this can not be considered as a center of the arrangement. Though a prism is bounded by five planes, there are several prisms, and none can be considered as the center.

When $m = 6$, we can not describe the arrangement in a few lines; we will see that there is no center of the arrangement. The last author made a study of this arrangement in [CYY], but he himself admits the insufficiency of the description. To make the scenery of this arrangement visible, we spend more than thirty pages. The chamber bounded by six faces which is not a cube plays an important role: this is bounded by two pentagons, two triangles and two quadrilaterals. This fundamental chamber seems to have no name yet. So we name it as a *dumpling* (*gyoza* in [CY, CYY]).

We study arrangements of m hyperplanes in the real projective n -space in general position. As we wrote already, there would be no interesting things in general. So we restrict ourselves to consider the *Veronese arrangements* (this will be defined in the text); when $n = 2$, it means that m lines bound an m -gon. In general such an arrangement can be characterized by the existence of the action of the cyclic group $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ of order m . Note the fact: if $m \leq n + 3$ then every arrangement is Veronese.

When n is even, there is a unique chamber which is stable under the group; this chamber will be called the *central* one. When $n = 0$, it is the unique point, and when $n = 2$, it is the m -gon. We are interested in the next case $n = 4$, which we study in detail. In particular, when $(n, m) = (4, 7)$, the central chamber is bounded by seven dumplings. This study gives a light to general even dimensional central chamber. Remember that central chambers will be higher dimensional versions of a point and a pentagon (and an m -gon).

When n is odd, other than the case $(n, m) = (3, 6)$, we study higher dimensional versions of a dumpling. Note that the 1-dimensional dumpling should be an interval.

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1 Preliminaries

We consider **arrangements** of m hyperplanes in the n -dimensional real projective space

$$\mathbb{P}^n := \mathbb{R}^{n+1} - \{0\} \text{ modulo } \mathbb{R}^\times.$$

\mathbb{P}^{-1} is empty, \mathbb{P}^0 is a singleton, \mathbb{P}^1 is called the projective line, \mathbb{P}^2 the projective plane, and \mathbb{P}^n the projective n -space. We always suppose the hyperplanes are *in general position* (i.e. no $n + 1$ hyperplanes meet at a point). We often work in the n -dimensional sphere

$$\mathbb{S}^n := \mathbb{R}^{n+1} - \{0\} \text{ modulo } \mathbb{R}_{>0},$$

especially when we treat inequalities. This is just the double cover of \mathbb{P}^n , which is obtained by identifying the anti-podal points of \mathbb{S}^n . So, a hyperplane in \mathbb{S}^n is nothing but a punctured vector hyperplane modulo $\mathbb{R}_{>0}$.

1.1 Projective spaces

Let us give some geometric idea of the projective spaces.

- Two distinct hyperplanes meet along a projective space two dimensional lower. Two distinct lines in \mathbb{P}^2 meet at a point. Two distinct points in \mathbb{P}^1 do not meet.

- If you think a hyperplane as the hyperplane at infinity, what remains is the usual euclidean space. A tubular neighborhood of a line – the complement of a disc – in \mathbb{P}^2 is a Möbius strip.
- If you keep walking along a straight line then you eventually come back along the same line from behind to the point you started.
- Two parallel lines meet at a point at infinity.
- \mathbb{P}^1 as well as \mathbb{S}^1 is just a circle.
- \mathbb{P}^2 can not be embedded into \mathbb{R}^3 unless you permit a self-intersection.
- Odd dimensional ones are orientable, and even dimensional ones are non-orientable.

1.2 Hyperplane arrangements

We consider arrangements of m hyperplanes in \mathbb{P}^n . We always assume that an arrangement is *in general position*, which means no $n + 1$ hyperplanes meet at a point. Let $x_0 : x_1 : \dots : x_n$ be a system of homogeneous coordinates on \mathbb{P}^n . A hyperplane H is defined by a linear equation

$$a_0x_0 + a_1x_1 + \dots + a_nx_n = 0, \quad (a_0, \dots, a_n) \neq (0, \dots, 0).$$

By corresponding a hyperplane H its coefficients $a_0 : \dots : a_n$ of the defining equation, we have an isomorphism between the set of hyperplanes and the set of points in the dual projective space (i.e. a -space). A hyperplane arrangement is *in general position* if and only if the corresponding point arrangement is in general position, which means no $n + 1$ points are on a hyperplane. By definition an arrangement of m hyperplanes H_1, \dots, H_m , is defined up to the action of the symmetry group on the indices $1, \dots, m$.

1.2.1 Grassmann isomorphism

An arrangement of $m(\geq n + 2)$ hyperplanes

$$H_j : a_{0j}x_0 + a_{1j}x_1 + \dots + a_{nj}x_n = 0, \quad j = 1, \dots, m$$

defines an $(n + 1) \times m$ -matrix $A = (a_{ij})$. We can regard A as a matrix representing a linear map from a linear space of dimension m to that of dimension $n + 1$. Note that the arrangement is in general position if and only if no $(n + 1)$ -minor vanish. Choose a basis of the kernel of this map and arrange them vertically, and we get an $m \times (m - n - 1)$ -matrix B such that $AB = 0$. The matrix tB defines an arrangement of m hyperplanes in \mathbb{P}^{m-n-2} . The choice of the bases is not unique; the ambiguity gives projective transformations of \mathbb{P}^{m-n-2} . No $(m - n - 1)$ -minor of B vanish (linear (in)dependence of the first $n + 1$ columns of A implies that of the last $m - n - 1$ lines of B).

Summing up, we get an isomorphism between the arrangements of m hyperplanes in \mathbb{P}^n and those in \mathbb{P}^{m-n-2} .

1.2.2 Arrangements when $m = n + 2$

It is well-known that any three distinct points on the projective line $\mathbb{P}^1 = \mathbb{R} \cup \{\infty\}$ can be transformed projectively into $\{0, 1, \infty\}$. In general, we want to prove that any two systems of $n + 2$ points in general position in \mathbb{P}^n can be transformed projectively to each other. Since Grassmann isomorphism doesn't make sense when $m - n - 2 = 0$, we prove this fact using linear algebra.

Proposition 1 *Any $n + 2$ points in general position in \mathbb{P}^n can be transformed projectively into the $n + 2$ points:*

$$1 : 0 : \cdots : 0, \quad 0 : 1 : 0 : \cdots : 0, \quad \dots, \quad 0 : \cdots : 0 : 1, \quad 1 : \cdots : 1.$$

proof: Let A be the corresponding $(n + 1) \times (n + 2)$ -matrix. Multiplying a suitable matrix from the left, we can assume that A is of the form $(I_{n+1} \ a)$, where $a = {}^t(a_0, \dots, a_n)$. Since the arrangement is in general position, $a_j \neq 0$ ($j = 0, \dots, n$). We then multiply $\text{diag}(1/a_0, \dots, 1/a_n)$ from the left.

Projective transformations still operate on these points as permutations of $n + 2$ points.

1.2.3 Arrangements when $m = n + 3$

Proposition 2 *The set of arrangements of $n + 3$ hyperplanes in general position in \mathbb{P}^n is connected.*

We can use the Grassmann isomorphism (note that $m - n - 2 = 1$). The arrangements of m points in \mathbb{P}^1 is topologically unique, but not projectively of course. We give direct proof without using Grassmann isomorphism. We prove the dual statement: The set of arrangements of $n + 3$ points in \mathbb{P}^n in general position is connected.

proof: Put $n + 2$ points as in Proposition 1. Where can we put the $(n + 3)$ -th point so that the $n + 3$ points are in general position, that is, no $n + 1$ points are on a hyperplane? n points out of these $n + 2$ points span hyperplanes defined by:

$$x_i = 0, \quad x_j = x_k \quad (i, j, k = 1, \dots, n + 1, \ j \neq k).$$

These hyperplanes are places which are not allowed to put the $(n + 3)$ -th point. These hyperplanes divide the space \mathbb{P}^n into simplices (if non-empty) defined by

$$x_{i_1} < x_{i_2} < \cdots < x_{i_{n+1}}, \quad \{i_1, \dots, i_{n+1}\} \subset \{0, 1, \dots, n + 1\},$$

where $x_0 = 0, x_{n+1} = 1$. The symmetric group on $n + 2$ letters acts transitively on these simplices. This completes the proof.

We can rephrase these propositions as:

Corollary 1 *If $m \leq n + 2$ there is only one arrangement up to linear transformations. If $m = n + 3$ there is only one arrangement up to continuous move keeping the intersection pattern.*

1.2.4 Curves of degree n

Consider a curve $x(t)$ in \mathbb{P}^n given by

$$x_0 = x_0(t), \quad x_1 = x_1(t), \quad \dots, \quad x_n = x_n(t), \quad t \in \mathbb{R}.$$

At $t = \tau$, if there is a hyperplane

$$y_0(\tau)x_0 + y_1(\tau)x_1 + \dots + y_n(\tau)x_n = 0$$

passing through $x(\tau)$ such that the derived vectors $x'(\tau), x''(\tau), \dots, x^{(n-1)}(\tau)$ lie on it, that is,

$$X(\tau) \begin{pmatrix} y_0(\tau) \\ y_1(\tau) \\ \vdots \\ y_n(\tau) \end{pmatrix} = 0, \quad X(\tau) = \begin{pmatrix} x_0(\tau) & x_1(\tau) & \dots & x_n(\tau) \\ x'_0(\tau) & x'_1(\tau) & \dots & x'_n(\tau) \\ \dots & \dots & \dots & \dots \\ x_0^{(n-1)}(\tau) & x_1^{(n-1)}(\tau) & \dots & x_n^{(n-1)}(\tau) \end{pmatrix},$$

this hyperplane is called an *osculating hyperplane* of the curve at $x(\tau)$. If the rank of $X(\tau)$ is n , there is a unique osculating hyperplane at $x(\tau)$. The correspondence

$$\tau \longmapsto y_0(\tau) : y_1(\tau) : \dots : y_n(\tau)$$

defines a curve in the dual projective space (i.e. y -space). Notice that (since $(\sum x_k y_k)' = \sum x'_k y_k + \sum x_k y'_k$, $(\sum x'_k y_k)' = \sum x''_k y_k + \sum x'_k y'_k$, etc) we have

$$Y(\tau) \begin{pmatrix} x_0(\tau) \\ \vdots \\ x_n(\tau) \end{pmatrix} = 0, \quad Y(\tau) = \begin{pmatrix} y'_0(\tau) & \dots & y'_n(\tau) \\ y_0(\tau) & \dots & y_n(\tau) \\ \dots & \dots & \dots \\ y_0^{(n-1)}(\tau) & \dots & y_n^{(n-1)}(\tau) \end{pmatrix},$$

that is, the hyperplane

$$x_0(\tau)y_0 + \dots + x_n(\tau)y_n = 0$$

in y -space is an osculating hyperplane of the curve $y(t)$ at $y(\tau)$. This curve $y(t)$ is called the *dual curve*. Notice also that if $x(t)$ is of degree n , so is $y(\tau)$. Thanks to this dual correspondence, in place of proving

Proposition 3 *For any $n + 3$ hyperplanes in general position in \mathbb{P}^n , there is a unique rational curve of degree n osculating these hyperplanes.*

we prove

Proposition 4 *For any $n + 3$ points in general position in \mathbb{P}^n , there is a unique rational curve of degree n passing through these points.*

This is a generalization of a well-known fact that there is a unique conic passing through given five points in general position.

Without loss of generality, we put $n + 3$ points as:

$$\begin{aligned}
& x_1 : x_2 : \cdots : x_n : x_{n+1} \\
& p_0 = 1 : 1 : \cdots : 1 : 1, \\
& p_1 = 1 : 0 : \cdots : 0 : 0, \\
& p_2 = 0 : 1 : \cdots : 0 : 0, \\
& \vdots \\
& p_n = 0 : 0 : \cdots : 1 : 0, \\
& p_{n+1} = 0 : 0 : \cdots : 0 : 1, \\
& p_{n+2} = a_1 : a_2 : \cdots : a_n : a_{n+1},
\end{aligned}$$

where $0 < a_1 < \cdots < a_n < a_{n+1}$. Sorry, in this proof we use coordinates $x_1 : \cdots : x_{n+1}$ instead of $x_0 : \cdots : x_n$. We will find a curve

$$C : t \longmapsto x_1(t) : \cdots : x_{n+1}(t),$$

such that $x_j(t)$ is a polynomial in t of degree n , and that

$$C(q_0) = p_0, \quad C(q_1) = p_1, \quad \dots, \quad C(q_{n+1}) = p_{n+1}, \quad C(r) = p_{n+2}.$$

If we normalize as

$$q_0 = \infty, \quad q_1 = 0, \quad q_2 = 1,$$

then the above condition is equivalent to the system of equations

$$\begin{aligned}
(x_1(r) =) \quad & c(r - q_2)(r - q_3)(r - q_4) \cdots (r - q_{n+1}) = a_1, \\
(x_2(r) =) \quad & c(r - q_1)(r - q_3)(r - q_4) \cdots (r - q_{n+1}) = a_2, \\
(x_3(r) =) \quad & c(r - q_1)(r - q_2)(r - q_4) \cdots (r - q_{n+1}) = a_3, \\
& \vdots \\
(x_{n+1}(r) =) \quad & c(r - q_1)(r - q_2)(r - q_4) \cdots (r - q_n) = a_{n+1},
\end{aligned}$$

with $n + 1$ unknowns q_3, \dots, q_{n+1}, r and c . From the first and the second equations, r is solved, from the second and the third equation, q_3 is solved, ..., and we obtain a unique set of solutions:

$$r = \frac{a_2}{a_2 - a_1}, \quad q_j = \frac{(a_j - a_1)a_2}{(a_2 - a_1)a_j} \quad (j = 3, \dots, n + 1)$$

We do not care the value of c . Since

$$q_3 - 1 = \frac{(a_3 - a_2)a_1}{(a_2 - a_1)a_3}, \quad q_j - q_i = \frac{a_1 a_2}{a_2 - a_1} \cdot \frac{a_j - a_i}{a_j a_i}, \quad r - q_j = \frac{a_1 a_2}{a_2 - a_1} \cdot \frac{1}{a_j}$$

we have

$$q_1 = 0 < q_2 = 1 < q_3 < \cdots < q_{n+1} < r.$$

1.3 Number of chambers

Let $\#(n, m)$ be the number of chambers cut out from the projective n -space \mathbb{P}^n by m hyperplanes in general position. We have

$$\#(1, m) = m, \quad \#(n, 1) = 1.$$

Consider $m - 1$ hyperplanes in \mathbb{P}^n . The m -th hyperplane intersects $m - 1$ hyperplanes, which cut out $\#(n - 1, m - 1)$ chambers of dimension $n - 1$. Each $(n - 1)$ -dimensional chamber cut an n -dimensional chamber into two. Thus we have

$$\#(n, m) = \#(n, m - 1) + \#(n - 1, m - 1),$$

and so

$$\#(n, m) = 1 + \sum_{\ell=1}^{m-1} \#(n - 1, \ell).$$

This can be readily solved:

$$\#(n, m) = \binom{m}{n} + \binom{m}{n-2} + \cdots + \binom{m}{2} + \binom{m}{0}, \quad n : \text{even},$$

$$\#(n, m) = \binom{m}{n} + \binom{m}{n-2} + \cdots + \binom{m}{3} + \binom{m}{1}, \quad n : \text{odd},$$

where $\binom{m}{n} = 0$ if $m < n$. In particular, $\#(1, m) = m$,

$$\#(2, m) = \binom{m}{2} + 1, \quad \#(3, m) = \binom{m}{3} + m, \quad \#(4, m) = \binom{m}{4} + \binom{m}{2} + 1.$$

1.4 Rotating a polytope around a face

Let P be an n -polytope in \mathbb{R}^n canonically embedded in \mathbb{R}^{n+1} , and σ one of its facets $((n - 1)$ -dimensional faces). The convex hull of P and its image by a rotation in \mathbb{R}^{n+1} of an angle less than π and centered at σ is an $(n + 1)$ -polytope, which will be denoted by P^σ , and will be called a polytope obtained from P by *rotating* it around σ . It can be abstractly defined as

$$P^\sigma = P \times [0, 1] / \sim, \quad (x, t) \sim (x, 0), \quad x \in \sigma, \quad t \in [0, 1],$$

in other words it is obtained by crushing the facet $\sigma \times [0, 1]$ of the product $P \times [0, 1]$ by the projection (see Figure 1)

$$\sigma \times [0, 1] \ni (x, t) \mapsto x \in \sigma.$$

Note that the boundary of P^σ consists of

$$P \times \{0\}, \quad P \times \{1\}, \quad (\tau \times [0, 1]) / \sim \quad (\tau : \text{face of } P, \tau \neq \sigma).$$

For example, rotating a pentagon around a side, we get a polytope bounded by two pentagons, two triangles, and two quadrilaterals. In other words, it is obtained by crushing a rectangular face of the pentagonal prism. As is shown in Figure 1 (right), it looks like a dumpling, so it will be called a **dumpling**. This is called a *gyoza* in [CY, CYY]. In general, rotating an m -gon around a side, we get a polytope bounded by two m -gon's, two triangles, and $m - 3$ quadrilaterals; this polytope will also be called a (3-dimensional m -) dumpling. A 3-dumpling is a tetrahedron, a 4-dumpling is a triangular prism.

High dimensional ones will be also called *dumplings*.

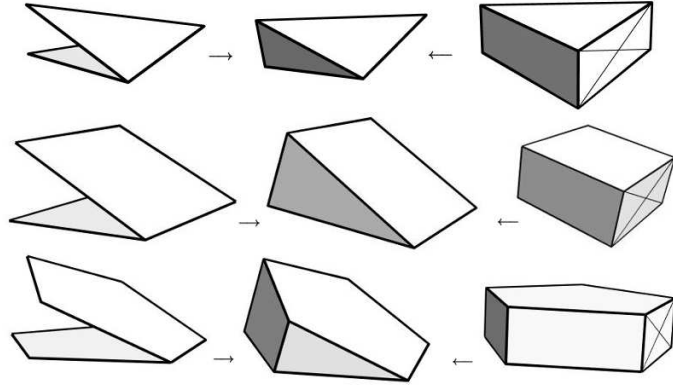


Figure 1: Dumplings $n = 3, m = 3, 4, 5$

2 Veronese arrangements

A hyperplane arrangement $A = \{H_j\}_{j=1,\dots,m}$ in \mathbb{S}^n (or \mathbb{P}^n) is said to be **Veronese** if under a suitable linear change of coordinates, H_j is given by $f_j = 0$, where

$$f_j = f(t_j, x) = x_0 + t_j x_1 + t_j^2 x_2 + \cdots + t_j^n x_n, \quad j = 1, \dots, m.$$

Here $t_1 < t_2 < \cdots < t_m$ are real numbers, and (x_0, \dots, x_n) are coordinates on \mathbb{R}^{n+1} . Note that by proposition 3, if $m \leq n + 3$, every arrangement of hyperplanes in general position is Veronese.

Proposition 5 *For the curve V_n of degree n defined by*

$$\mathbb{R} \ni t \mapsto x(t) = (-t)^n : n(-t)^{n-1} : \cdots : \binom{n}{k} (-t)^{n-k} : \cdots : 1,$$

the osculating hyperplane at the point $x(t_j)$ is given by $f_j = 0$.

The curve V_n is often called a Veronese embedding of \mathbb{P}^1 into \mathbb{S}^n (or \mathbb{P}^n). This is the reason for coining Veronese arrangements.

From here to the end of next section, we work in \mathbb{S}^n . The binomial theorem tells

$$f_j(x(t)) = (t_j - t)^n;$$

so that if n is even,

$$f_j(x(t)) > 0 \quad \text{if} \quad t \neq t_j,$$

if n is odd,

$$f_j(x(t)) < 0 \quad \text{if} \quad t < t_j, \quad f_j(x(t)) > 0 \quad \text{if} \quad t > t_j.$$

2.1 Chambers cut out by Veronese arrangements

The closure of each connected component of the complement of a hyperplane arrangement is called a **chamber**. Let a Veronese arrangement be given as in the previous section. Each chamber is given by a system of inequalities

$$\varepsilon_j f_j \geq 0, \quad j = 1, \dots, m, \quad (\varepsilon_j = \pm 1)$$

which is often denoted by the sequence

$$\varepsilon_1, \dots, \varepsilon_m.$$

Group action: Since the Veronese arrangement is determined by a sequence t_1, t_2, \dots, t_m of points in \mathbb{P}^1 arranged in this order, the shift

$$\iota : j \mapsto j + 1 \pmod{m}$$

acts on the set of chambers by

$$\varepsilon_1, \dots, \varepsilon_m \mapsto \begin{cases} -\varepsilon_m, \varepsilon_1, \dots, \varepsilon_{m-1} & m : \text{odd}, \\ \varepsilon_m, \varepsilon_1, \dots, \varepsilon_{m-1} & m : \text{even}. \end{cases}$$

The cyclic group generated by ι will be denoted by \mathbb{Z}_m . Note that if the points t_1, t_2, \dots, t_m are so arranged that the transformation $t_j \rightarrow t_{j+1}$ is given by a projective one, then the corresponding action on \mathbb{P}^n is also projective. Thus we have the well-defined \mathbb{Z}_m action on the set of chambers as above for any sequence $t_1, t_2, \dots, t_m \in \mathbb{P}^1$.

2.2 A specific feature of Veronese arrangements

Let a Veronese arrangement $A = \{H_j\}$ of m hyperplanes

$$H_j : f_j = f(t_j, x) = 0, \quad \dots < t_j < t_{j+1} < \dots$$

in \mathbb{P}^n be given as in the previous section. If we let t_j tends near to t_{j+1} , the intersection pattern does not change, that is, there is no vertex (intersection point of n hyperplanes in A) in the slit between the hyperplanes H_j and H_{j+1} . More precisely, the hyperplanes H_j and H_{j+1} divide the space into two parts, and one of them does not contain any vertex.

This fact gives the following information on the chambers cut out by A . If a chamber in the slit between the hyperplanes H_j and H_{j+1} does not touch the intersection $H_j \cap H_{j+1}$, then this chamber is the direct product of a chamber of the restricted arrangement A_j and the unit interval. If a chamber in the slit touches the intersection of the two hyperplanes, then (since the hyperplanes are in general position) there is a chamber P of the restricted arrangement A_j with a facet σ included in $H_j \cap H_{j+1}$ and the chamber in the slit is the polyhedron P^σ obtained from P by rotating around σ .

A polyhedron (of dimension greater than 1) is said to be *irreducible* if it is not a direct product of two polyhedron nor is obtained from a lower one by rotation described in §1.4. For example, triangles, rectangles, tetrahedra, prisms, cubes and dumplings are reducible, while pentagon is irreducible.

A combinatorial study on Veronese arrangements is made in [CY]. We do not use in this paper the result obtained there. To give a general idea, we just quote one of the results: Consider a Veronese arrangement $A(m, n)$ of m ($\geq n + 3$) hyperplanes in \mathbb{P}^n ($n \geq 2$).

- If n is odd, then every chamber is a direct product of the unit interval and a chamber of $A(m - 1, n - 1)$, or is obtained from a chamber of $A(m - 1, n - 1)$ by rotating it around a facet.

- If n is even, then there is a unique irreducible chamber. Other chamber is obtained from $A(m-1, n-1)$ like the above.

In this paper, for a Veronese arrangement of even dimension, the unique irreducible chamber will be called the *central chamber*.

3 Six planes in the 3-space

In this case there is no central chamber. The arrangement is Veronese. We are going to choose a particular arrangement so that the action of a group $G \cong \mathfrak{S}_3 \times \mathbb{Z}_2$ of orientation preserving projective transformations will be geometrically visible.

The elements of the group are given by matrices (up to scalar multiplication) with coefficients 0, 1, -1 . The group G is a subgroup of $PGL_+(4, \mathbb{Z})$. Notice that $GL_+(4, \mathbb{Z})$ is not a group since inverse operation is not always defined. However it is defined up to scalar matrix multiplication, so that the quotient $PGL_+(4, \mathbb{Z})$ has a group structure.

The group G acts on the 26 chambers with four orbits: 12 prisms, 6 tetrahedra, 6 dumplings, 2 cubes. The orbit of a prism together with a cube is a solid torus. The orbit of a dumpling is a second solid torus linked with the first one. The orbit of a tetrahedron is the complementary of the two solid tori. Actually, the union of one of these solid tori and the orbit of a tetrahedron is a solid torus as well.

3.1 Simple observations

3.1.1 Cutting a tetrahedron and a prism

Chambers in the plane bounded by five or less lines are

diangle, triangle, quadrilateral, and pentagon.

Chambers in the space bounded by four or less planes are dihedron, trihedron, tetrahedron; to see further we cut a tetrahedron by a plane. There are two ways to cut (see Figure 2). If we denote a tetrahedron by T , and a (triangular) prism by P , then the two cuttings

Figure 2: Cutting a tetrahedron (tetracut)

can be presented as

$$T \rightarrow T + P \quad (\text{triangle}), \quad P + P \quad (\text{quadrilateral}),$$

where $T + P$ (triangle) means T and P share a triangle.

We cut next a prism. There are five ways to cut (see Figure 3). If we denote a cube by C , and a (pentagonal) dumpling by D , then the five cuttings can be presented as

$$\begin{aligned} T + D \quad (\text{triangle}), \quad P + D \quad (\text{quadrilateral}), \quad P + C \quad (\text{quadrilateral}), \\ P + P \quad (\text{triangle}), \quad D + D \quad (\text{pentagon}). \end{aligned}$$

The last one will appear again in §4.3.4.

Figure 3: Cutting a prism (prismcut)

3.1.2 A few facts seen from the above cuttings

Since five planes cut out tetrahedra and prisms, if two chambers cut out by an arrangement of six planes are adjacent along a face, the union is a tetrahedron or a prism. Thus though C and D have quadrilateral faces, they are not face to face.

When two D 's are face to face along a pentagon, the remaining two pentagons do not share an edge. Since there is one pentagon on each plane (see Introduction), and since the group \mathbb{Z}_6 acts on the arrangement, we conclude that there are six D 's, which are glued to form a solid torus (see §3.4).

3.2 Setting

If we can choose coordinates on \mathbb{P}^n and the equations of $n + 3$ hyperplanes so nicely that the \mathbb{Z}_{n+3} -action is clearly seen, it would be nice. Though we can not expect this in general, when $n = 3$, there is a very good choice ([Mo] and remark after proposition 6). We work in the real projective space coordinatized by $x : y : z : t$. Our six planes are

$$\begin{aligned} H_x : x = 0, \quad H_y : y = 0, \quad H_z : z = 0, \\ H^x : h^x = 0, \quad H^y : h^y = 0, \quad H^z : h^z = 0, \end{aligned}$$

where

$$h^x := y - z - t, \quad h^y := z - x - t, \quad h^z := x - y - t.$$

Note that if we put $h^t = x + y + z$, then we have

$$h^y - h^z - h^t = -3x, \quad h^z - h^x - h^t = -3y, \quad h^x - h^y - h^t = -3z.$$

Note also that if we change from t to $-t$, the new arrangement is the mirror image of the original one.

We often work in the Euclidean 3-space coordinatized by

$$(x, y, z) = x : y : z : 1,$$

especially when we speak about distance and/or angle, without saying so explicitly.

3.2.1 A group action and a cubic curve K

The six planes admit the transformations

$$\begin{aligned} \rho : H_x \rightarrow H_y \rightarrow H_z \rightarrow H_x, \quad H^x \rightarrow H^y \rightarrow H^z \rightarrow H^x, \\ \sigma : H_x \rightarrow H_y, H_y \rightarrow H_x, H_z \rightarrow H_z, \quad H^x \rightarrow H^y, H^y \rightarrow H^x, H^z \rightarrow H^z, \quad \text{and} \\ \tau : H_a \rightarrow H^a \rightarrow H_a, \quad (a = x, y, z) \end{aligned}$$

which are of order 3, 2 and 2, respectively. These are given by the projective transformations

$$\rho = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & \bar{1} & 0 & 0 \\ \bar{1} & 0 & 0 & 0 \\ 0 & 0 & \bar{1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & 1 & \bar{1} & \bar{1} \\ \bar{1} & 0 & 1 & \bar{1} \\ 1 & \bar{1} & 0 & \bar{1} \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

where $\bar{1} = -1$. Note that $\tau^2 = -3I_4$. In the euclidean space, ρ acts as the $2\pi/3$ -rotation around the axis generated by the vector $(1, 1, 1)$, and σ the π -rotation about the axis generated by the vector $(1, -1, 0)$. The transformation τ exchanges the plane

$$H^\infty : x + y + z = 0$$

and the plane $H_\infty : t = 0$ at infinity. They generate the orientation preserving projective transformation group

$$G \cong \langle \rho, \sigma \rangle \times \langle \tau \rangle \cong \mathfrak{S}_3 \times \mathbb{Z}_2,$$

which is a subgroup of $PGL_+(4, \mathbb{Z})$ of order 12 with center $\langle \tau \rangle = \{1, \tau\}$. The relations $\sigma^2 = 1$, $(\rho\tau)^6 = 1$ and $(\rho\tau\sigma)^2 = 1$ show that G is also isomorphic to the dihedral group

$$D_6 \cong \langle \rho\tau \rangle \rtimes \langle \sigma \rangle \cong \mathbb{Z}_6 \rtimes \mathbb{Z}_2.$$

Proposition 6 *There is no invariant plane by G . There are two orbits of order two,*

$$\{H^\infty, H_\infty\} \quad \text{and} \quad \{H^{\sqrt{3}}, H_{\sqrt{3}}\},$$

where

$$H^{\sqrt{3}} : x + y + z - t\sqrt{3} = 0, \quad H_{\sqrt{3}} : x + y + z + t\sqrt{3} = 0.$$

There is no orbit of order three.

proof. If H is invariant by τ , it is different from H_∞ . If, in addition, it is invariant by ρ it is orthogonal (in the euclidean space) to the vector $(1, 1, 1)$. Such a plane is not invariant by σ , so that there is no plane invariant by G .

Suppose now that H belongs to an orbit of order 2, say $\{H, H'\}$, different from $\{H^\infty, H_\infty\}$. The group $\langle \rho \rangle \cong \mathbb{Z}_3$ acts necessarily trivially on a set of order 2 so that $\rho(H) = H$. Then, as above, H is orthogonal to the vector $(1, 1, 1)$ and is not invariant by σ so that $\sigma(H) = H'$. The equation of H writes $x + y + z - \alpha t = 0$, and the one of H' writes $x + y + z + \alpha t = 0$. Writing that $\tau(H) = H$ or $\tau(H) = H'$, we find that $\alpha = \pm\sqrt{3}$.

Now, suppose that H belongs to an orbit of order 3, then the order two elements σ and τ act trivially on this orbit, so that $\sigma(H) = \tau(H) = H$, and similarly for $\rho(H)$ and $\rho^2(H)$. In particular $H \neq H_\infty$ and, since σ is a half-turn rotation about the vector $(1, -1, 0)$, either H is orthogonal to the vector $(1, -1, 0)$ or H contains the line generated by the vector $(1, -1, 0)$.

In the first case, $\rho(H)$ is no longer orthogonal to the vector $(1, -1, 0)$ so that $\rho(H)$, and $\rho^2(H)$ as well, must contain the line generated by the vector $(1, -1, 0)$, so that $\rho(H)$,

and then H itself, is orthogonal to the vector $(1, 1, 1)$. Therefore H would be invariant, which is impossible.

In the second case, since H is not invariant and then not orthogonal to the vector $(1, 1, 1)$, the plane $\rho(H)$ doesn't contain the line generated by the vector $(1, -1, 0)$, and therefore is orthogonal to the vector $(1, -1, 0)$. We are brought back to the first case. \square

Each plane intersects the remaining five; they cut out a pentagon, five triangles and five rectangles. See Figures 5, and 4 for H_\bullet and H° , respectively.

The six planes form a Veronese arrangement in the order:

$$H_x, \quad H^y, \quad H_z, \quad H^x, \quad H_y, \quad H^z, \quad H_x,$$

by which we mean, there is a unique rational cubic curve K osculating these six planes in this order. Note that ρ and τ respect the order, and σ just reverses the order. Proof goes as follows.

If the regular hexagon (the projective line coordinatized by s) is given by

$$s = 0, \quad 1, \quad 3/2, \quad 2, \quad 3, \quad \infty,$$

(here, regular means there is a projective transformation $s \rightarrow 3/(3-s)$ sending 0 to 1, 1 to $3/2$, \dots , ∞ to 0) then the osculating curve is given by

$$K : x = s^3, \quad y = (s-3)^3, \quad z = -8(s-3/2)^3, \quad t = 9(s-1)(s-2),$$

which osculates the planes H_x, \dots, H^z at

$$K(0) = 0 : 3 : \bar{3} : 2, \quad K(1) = 1 : \bar{8} : 1 : 0, \quad K(\frac{3}{2}) = \bar{3} : 3 : 0 : 2,$$

$$K(2) = \bar{8} : 1 : 1 : 0, \quad K(3) = 3 : 0 : \bar{3} : 2, \quad K(\infty) = 1 : 1 : \bar{8} : 0. \quad \square$$

Remark. The chosen arrangement can be called **selfadjoint** in the following sense. The dual coordinates of the planes are

$$H_x = 1000, \quad H^y = \bar{1}01\bar{1}, \quad H_z = 0010, \quad H^x = 01\bar{1}\bar{1}, \quad H_y = 0100, \quad H^z = 1\bar{1}0\bar{1}.$$

For instance the equation of the plane $H^y = \bar{1}01\bar{1}$ writes $-x + z - t = 0$. The six planes turn out to be the points

$$P_x = 1000, \quad P^y = \bar{1}01\bar{1}, \quad P_z = 0010, \quad P^x = 01\bar{1}\bar{1}, \quad P_y = 0100, \quad P^z = 1\bar{1}0\bar{1}$$

in the dual space coordinatized by $\check{x} : \check{y} : \check{z} : \check{t}$. They are on the cubic curve \check{K} given by $\check{x} = 2(s-1)(s-3/2)(s-2)$, $\check{y} = -s(s-1)$, $\check{z} = (s-2)(s-3)$, $\check{t} = -3(s-1)(s-2)$, in this order. Indeed we have

$$\check{K}(\infty) = P_x, \quad \check{K}(0) = P^y, \quad \check{K}(1) = P_z, \quad \check{K}(3/2) = P^x, \quad \check{K}(2) = P_y, \quad \check{K}(3) = P^z.$$

The planes supported by three consecutive points

$$\langle P_x, P^y, P_z \rangle \quad \langle P^y, P_z, P^x \rangle \quad \langle P_z, P^x, P_y \rangle \quad \langle P^x, P_y, P^z \rangle \quad \langle P_y, P^z, P_x \rangle \quad \langle P^z, P_x, P^y \rangle$$

happen to be the original ones:

$$H_x, \quad H^y, \quad H_z, \quad H^x, \quad H_y, \quad H^z.$$

We sometimes code these six planes H_x, H^y, \dots as H_1, \dots, H_6 . Then the transformation of \mathbb{P}^1 above induces a projective transformation of \mathbb{P}^3 sending $H_j \rightarrow H_{j+1} \bmod 6$. This generates the cyclic group $\langle \rho, \tau \rangle \cong \mathbb{Z}_6$.

3.2.2 The twenty points

Convention: $H_{a\cdots}^{b\cdots} := H_a \cap \cdots \cap H^b \cap \cdots$. Three planes meet at a point; there are $\binom{6}{3} = 20$ of them. They are divided into three G -orbits:

$$\begin{aligned}
& \bullet \quad P_0 = H_{xyz} = 0 : 0 : 0 : 1, \quad \bigcirc \quad P^0 = H^{xyz} = 1 : 1 : 1 : 0; \\
& \blacksquare \quad H_x^{yz} = 0 : \bar{1} : 1 : 1, \quad H_y^{zx} = 1 : 0 : \bar{1} : 1, \quad H_z^{xy} = \bar{1} : 1 : 0 : 1, \\
& \square \quad H_{yz}^x = 1 : 0 : 0 : 0, \quad H_{zx}^y = 0 : 1 : 0 : 0, \quad H_{xy}^z = 0 : 0 : 1 : 0; \\
& \bullet \quad H_{xy}^x = 0 : 0 : \bar{1} : 1, \quad H_{yz}^y = \bar{1} : 0 : 0 : 1, \quad H_{zx}^z = 0 : \bar{1} : 0 : 1, \\
& \circ \quad H_x^{xy} = 0 : 2 : 1 : 1, \quad H_y^{yz} = 1 : 0 : 2 : 1, \quad H_z^{zx} = 2 : 1 : 0 : 1; \\
& \bullet \quad H_{yx}^y = 0 : 0 : 1 : 1, \quad H_{zy}^z = 1 : 0 : 0 : 1, \quad H_{xz}^x = 0 : 1 : 0 : 1, \\
& \circ \quad H_y^{yx} = \bar{2} : 0 : \bar{1} : 1, \quad H_z^{zy} = \bar{1} : \bar{2} : 0 : 1, \quad H_x^{xz} = 0 : \bar{1} : \bar{2} : 1,
\end{aligned}$$

where $\bar{1} = -1, \bar{2} = -2$. Note that the distances from the vertices to the origin P_0 are either

$$\bullet : 0, \quad \circ : 1, \quad \blacksquare : \sqrt{2}, \quad \circ : \sqrt{5}, \quad \square, \bigcirc : \infty.$$

In the $1 \cdots 6$ -coding, the three orbits above are represented by

$$135, \quad 123, \quad 124,$$

respectively, where 135 stands for $H_1 \cap H_3 \cap H_5$.

3.2.3 An invariant quadratic form

While there are many G -invariant quadratic forms in (x, y, z, t) , if we ask the zero set passes through the six vertices marked \square and \blacksquare , then it is (up to scalar multiplication) given by

$$Q = xy + yz + zx + t^2.$$

The surface $Q = 0$ and the curve K do not meet: in fact, substituting the expression of the curve in s into Q , we have $-15(s^2 - 3s + 3)^3$.

By identifying the space \mathbb{P}^3 and its dual, we regard the cubic curve \check{K} live in our space, that is, the curve is defined by

$$x = 2(s - 1)(s - 3/2)(s - 2), \quad y = -s(s - 1), \cdots$$

Then this is the unique rational cubic curve passing through the six points marked \square and \blacksquare :

$$\check{K}(\infty) = H_{yz}^x, \quad \check{K}(0) = H_y^{zx}, \quad \check{K}(1) = H_{xy}^z, \quad \check{K}(3/2) = H_x^{yz}, \quad \check{K}(2) = H_{zx}^y, \quad \check{K}(3) = H_z^{xy}.$$

This curve \check{K} is on the surface $Q = 0$.

3.2.4 The twenty six chambers

The six planes in \mathbb{P}^3 cut out

two cubes C , twelve prisms P , six tetrahedra T , six dumplings D .

The cubic curve K stays in the dumplings, and osculates each pentagon. The quadratic surface $Q = 0$ lies in the union of the tetrahedra and the dumplings (see Figures 5, 4).

3.2.5 The six planes

For the planes H^x, H^y, H^z , the intersection with other planes are shown in Figure 4. Here TP, DP, \dots stand for $T \cap P, D \cap P, \dots$. If the plane H^x , defined by $z = y - 1$, is

Figure 4: The planes H° (planeH-)

coordinatized by (y, x) , then the dotted curve is the conic (a hyperbola)

$$Q = xy + y(y - 1) + (y - 1)x + 1 = \left(y - \frac{1}{2}\right) \left(y + 2x - \frac{1}{2}\right) + \frac{3}{4} = 0.$$

For the planes H_x, H_y, H_z , the intersection with other planes are shown in Figure 5. If the plane H_x , defined by $x = 0$, is coordinatized by (y, z) , then the dotted curve is the

Figure 5: The planes H_\bullet (planeH)

conic (hyperbola) $Q = yz + 1 = 0$, and the point \heartsuit is the point $K(0)$ where the curve K kisses the plane H_x .

3.2.6 Bottom and top

It is convenient to consider the triangle with the three vertices \blacksquare in the plane $H^\infty : x + y + z = 0$ as the ground floor, and the triangle with the three vertices \square in the plane H_∞ at infinity as the ceiling. The planes H^x, H^y, H^z are orthogonal to the ground. The intersection (lines) of the six planes with the floor and the ceiling are shown in Figure 6. The dotted curves are the intersections with the surface $Q = 0$.

Figure 6: Bottom and top (bottom-top)

3.2.7 Octant chambers

The finite space coordinatized by

$$(x, y, z) = x : y : z : 1$$

is divided by the three planes H_x, H_y and H_z into eight chambers. We denote them as

$$(+++)=\{x \geq 0, y \geq 0, z \geq 0\}, \quad (++-)=\{x \geq 0, y \geq 0, z \leq 0\}, \dots$$

In this section, we see how these chambers are cut by the three planes H^x, H^y and H^z . In the chambers $(+++)$ and $(---)$, the happenings are similar, and in the remaining six chambers, similar things happen. Set

$$(+++; ---) = \{(x, y, z) \in (+++) \mid h^x \leq 0, h^y \leq 0, h^z \leq 0\}, \dots$$

If we write these as $(\varepsilon_1 \varepsilon_2 \varepsilon_3; \eta_1 \eta_2 \eta_3)$, they permit the \mathbb{Z}_3 -action

$$(\varepsilon_1 \varepsilon_2 \varepsilon_3; \eta_1 \eta_2 \eta_3) \rightarrow (\varepsilon_2 \varepsilon_3 \varepsilon_1; \eta_2 \eta_3 \eta_1).$$

In the chamber $(+++)$: There are four \mathbb{Z}_3 -orbits represented by

$$\begin{aligned} (+++; +++) &= \emptyset, & (\dot{+} \dot{+} \dot{+}; \dot{-} \dot{-} \dot{-}) &: \text{cube} \\ (\dot{+} \dot{+} \dot{+}; \dot{+} \dot{-} \dot{-}) &: \text{pri } P', & (+ \dot{+} \dot{+}; - \dot{+} \dot{+}) &: \text{sm } P''. \end{aligned}$$

Here effective ones are marked by dots. For example, the last one is defined by $y \geq 0, h^y \geq 0$ and $h^z \geq 0$; these three inequalities imply the other ones $x \geq 0, z \geq 0$ and $h^x \leq 0$. The number of dots corresponds to the number of walls.

As a whole, there are a cube and three P' and three P'' (see Figure 8). One P' and another P'' in the opposite chamber $(---)$ are glued along the plane at infinity forming a (full) prism (see Figure 7).

Figure 7: Unbounded prism $P = P' \cup P''$ (infprism80)

If we cut this chamber by a big sphere centered at the origin, the intersection with the chamber is surrounded by three arcs ($x = 0, y = 0, z = 0$), and the arc-triangle is cut by the three lines ($h^x = 0, h^y = 0, h^z = 0$). Around a triangle (cube), there are a triangle (P''), a pentagon (P'), a triangle (P''),... (see Figure 23 and Figure 24 left).

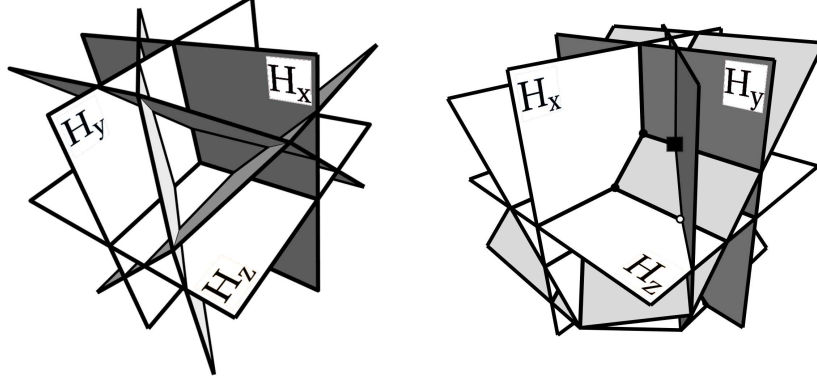


Figure 8: Chambers $(+ + +)$ and $(+ + -)$ (octants)

In the chamber $(- - -)$: There are four \mathbb{Z}_3 -orbits represented by

$$\begin{aligned} (- - -; + + +) &= \emptyset, & (\dot{-} \dot{-} \dot{-}; \dot{-} \dot{-} \dot{-}) &: \text{cube} \\ (\dot{-} \dot{-} \dot{-}; \dot{+} \dot{+} \dot{+}) &: \text{pri } P', & (- - \dot{-}; - \dot{+} \dot{+}) &: \text{sm } P''. \end{aligned}$$

In the chamber $(+ + -)$: This chamber does not admit the group action. Since $x \geq 0$ and $z \leq 0$ imply $z - x \leq 0$, and so $h^y = z - x - 1 < 0$, we have $(+ + -; * + *) = \emptyset$. Among the remaining four, there is a unique compact one:

$$(\dot{+} \dot{+} \dot{-}; \dot{-} - \dot{-}) : \text{prism.}$$

(See Figures 8(right) and 9(left).) The others are

$$(\dot{+} \dot{+} \dot{-}; \dot{-} - \dot{+}) : \text{tetrahedron,}$$

and

$$(\dot{+} \dot{+} \dot{-}; \dot{+} - \dot{-}) : \text{dump } D', \quad (\dot{+} \dot{+} \dot{-}; \dot{+} - \dot{+}) : \text{ling } D''.$$

One D' (resp. D'') and another D'' (resp. D') in the opposite chamber $(- - +)$ are glued along the plane at infinity forming a (full) dumpling (see Figure 9).

If we cut this chamber by a big sphere centered at the origin, the chamber is surrounded by three arcs ($x = 0, y = 0, z = 0$), and the arc-triangle is cut by the two lines ($h^x = 0, h^z = 0$). There are a triangle (tetrahedron) and two quadrilaterals, which are sections of D' and D'' (see Figure 23 and Figure 24 right).

In the chamber $(- - +)$: Since $y \leq 0$ and $z \geq 0$ imply $y - z \leq 0$, and so $h^x = y - z - 1 < 0$, we have $(- - +; + * *) = \emptyset$. Among the remaining four, there is a unique compact one:

$$(\dot{-} \dot{-} \dot{+}; - \dot{-} \dot{-}) : \text{prism.}$$

The others are

$$(\dot{-} - \dot{+}; - \dot{-} \dot{+}) : \text{tetrahedron,}$$

and

$$(\dot{-} \dot{-} \dot{+}; - \dot{+} \dot{-}) : \text{dump } D', \quad (\dot{-} - \dot{+}; - \dot{+} \dot{+}) : \text{ling } D''.$$

Figure 9: (right) A dumping D is cut by the plane at infinity into two parts dump D' and ling D'' . (left) They are in octants of type $(++-)$. The faces of D have names $1' + 1'', 2, 3' + 3'', 4, 5, 6' + 6''$. In the octant, a wall is $6'$ seen from D' , and is $6''$ seen from D'' . (octant70)

3.3 The solid torus made of the two cubes and the twelve prisms

The two cubes kiss at the two vertices P_0 and P^0 , which are opposite vertices of each cube. Around the two cubes are twelve prisms, forming with the two cubes a solid torus. We explain how they are situated.

3.3.1 The two cubes and the six prisms around P_0

The two cubes and the six prisms around P_0 , form in the finite (Euclidean) space an infinitely long triangular cylinder called **the big-cylinder** and bounded by H^x, H^y, H^z . It is defined by the inequations (figure 10):

$$h^x \leq 0, \quad h^y \leq 0, \quad h^z \leq 0.$$

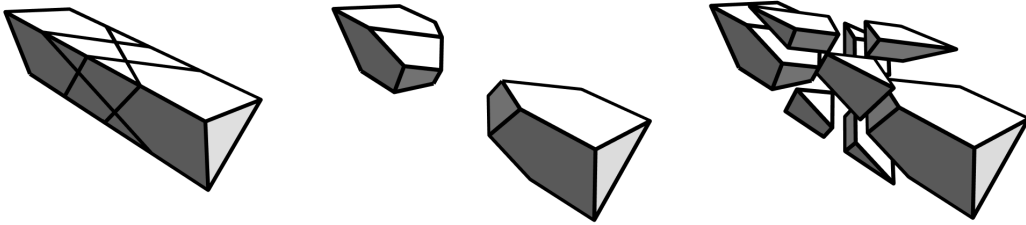


Figure 10: (left) The big-cylinder. (middle) The two truncated cubes slightly moved away from each other. (right) Exploded view of the two cubes and the six prisms.(amy-fig13.eps)

Part of the pictures in Figure 10 is enlarged in Figure 11.

3.3.2 The small cylinder

Considering the planes $H^{\sqrt{3}}$ and $H_{\sqrt{3}}$ defined in proposition 6, we have $\tau(H_{\sqrt{3}}) = H^{\sqrt{3}}$ where τ is the involution defined in 3.2.1. Let us denote by C_+ the heptahedron shown in figure 12 and defined by

$$\frac{x}{t} \geq 0, \quad \frac{y}{t} \geq 0, \quad \frac{z}{t} \geq 0, \quad \frac{x}{t} - \frac{y}{t} \leq 1, \quad \frac{y}{t} - \frac{z}{t} \leq 1, \quad \frac{z}{t} - \frac{x}{t} \leq 1, \quad \frac{x+y+z}{t} \leq \sqrt{3}.$$

We set $C_- := \sigma(C_+)$. The faces of the heptahedron C_+ are three triangles, three pentagons and one hexagon. We consider the solid SC (called **the small cylinder**) given by the

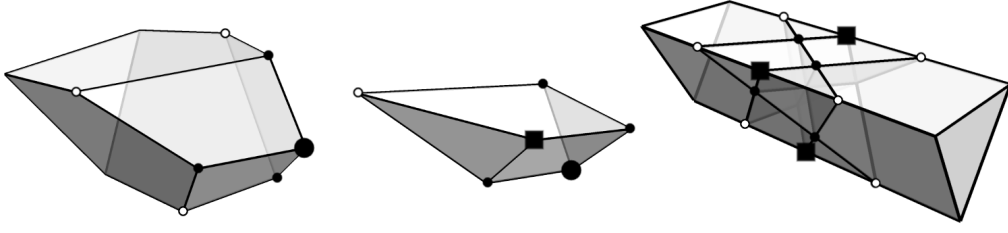


Figure 11: The big-cylinder made by two truncated cubes and six compact prisms (right). A truncated cube (left) and a prism (center) are shown with their vertices.

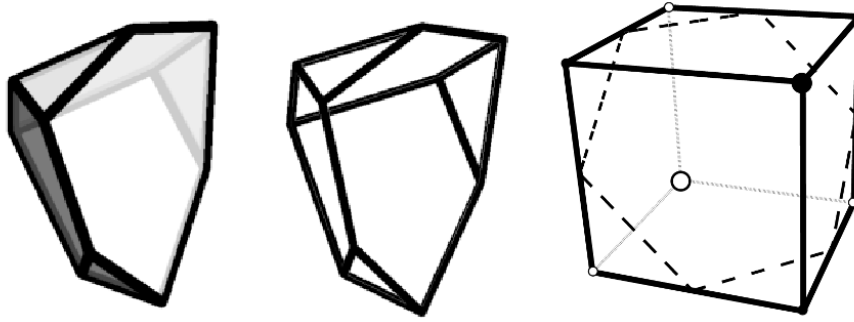


Figure 12: The heptahedron C_+ (left) and its 1-skeleton (center). Cutting a cube into two heptahedra (right)

union of the two heptahedrons and the six prisms. It is homeomorphic to a triangular prism by a homeomorphism preserving the generating lines (see figure 13).

3.3.3 The prism torus

The G -orbit of one of the six compact prisms around P_0 is given by twelve prisms, the six extra ones being located around P^0 . Now the image of the small cylinder SC by σ , or equivalently, the G -orbit of one prism and the heptahedron C_+ , form the union of the big cylinder and the six prisms around P^0 . These two solid cylinders SC and $\sigma(SC)$ are attached by σ along their bases and therefore form a solid torus. Let us call it the **prism torus**. Thus, this prism torus is decomposed into sixteen cells, four heptahedrons and twelve prisms, on which acts the group G .

3.3.4 The boundary of the prism torus

The prism torus consists also of the two cubes and the twelve prisms. Note that this torus is a tubular neighborhood of the projective line joining P_0 and P^0 . Therefore, as mentioned in the introduction of section 3, the complementary of this torus is a solid torus as well.

The above construction shows that the boundary of the torus consists of the faces of the prisms on the lateral boundary on the small cylinder SC and their images by σ . In particular, no face of the cubes belongs to the boundary of the prism torus.

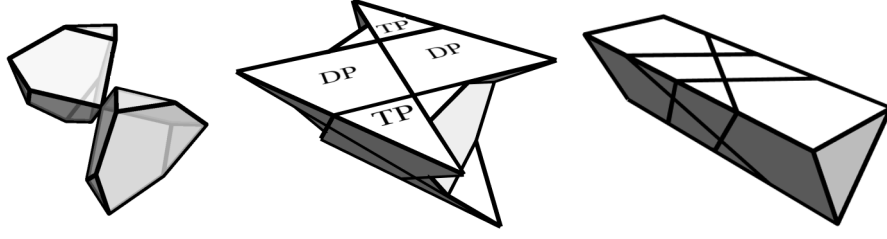


Figure 13: Heptahedrons C_+ and C_- kissing at P_0 (left). The small cylinder SC and two quadrilaterals DP and two triangles TP on its boundary (middle). The triangular prism homeomorphic to SC (right).

The boundary of the prism torus on each plane consists of two quadrilaterals DP and two triangles TP (see figure 13). They are shown more explicitly the pattern below:

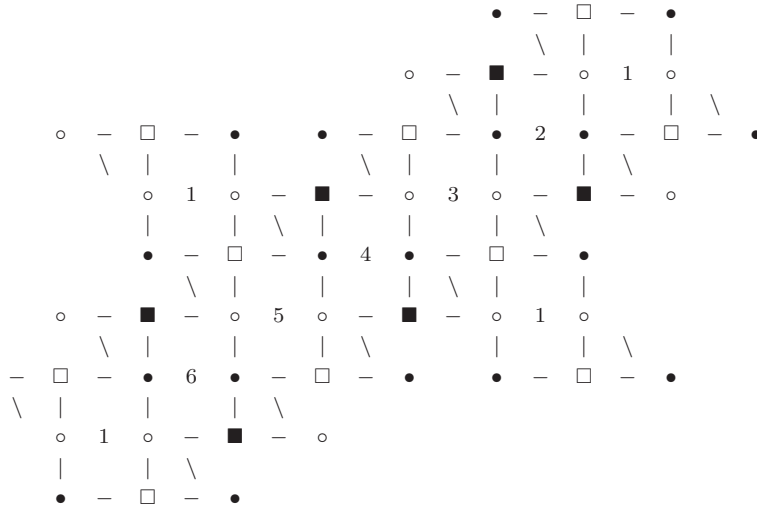


Figure 14: The boundary of the prism torus consists of the faces of the prisms (planeH2-70)

The twelve quadrilaterals and the twelve triangles tessellate the boundary torus as below. To indicate the identification, four copies of the plane number 1 are shown in addition. After identification, there are

$$3 \square, \quad 3 \blacksquare, \quad 6 \bullet, \quad 6 \circ;$$

note that among the twenty vertices, only P^0 and P_0 are missing.



3.4 The solid torus made of the six dumplings

Each of the six dumplings is glued along the two pentagonal faces with two others. The six dumplings form a solid torus; let us call it the **dumpling torus**. This section is devoted to understand it.

3.4.1 A dumpling

A dumpling has two rectangular faces, two triangular faces, and two pentagonal faces (see Figure 15), which share an edge with two vertices marked white and black squares. This edge will be called a **special edge** (see Figure 16).

Figure 15: Two kinds of pentagons bounding the dumplings (planeH3-70)

Figure 16: A dumpling, and two dumplings glued along a pentagon (gyoza170)

Each dumpling D is cut into two by the plane at infinity: the part with two vertices \bullet is denoted by D' , and the other by D'' . They are shown in Figure 9.

3.4.2 The dumpling torus

The six dumplings glued along their pentagonal faces form the dumpling torus. Figure 17 shows the six pentagons glued along the special edges forming a circle. In the figure, special edges are shown by thick segments.

Figure 17: Their pentagons glued along the special edges (pentagons70)

Remember that the planes

$$H_x, H^y, H_z, H^x, H_y, H^z, H_x \quad \text{are numbered} \quad 1, \dots, 6;$$

and the curve K lives inside the dumpling torus, and touches six pentagons in this order. In Figure 18, the six dumplings are glued along their pentagonal faces. The smallest pentagon on the top and the biggest one at the bottom are on the plane number 6; they should be identified after $2\pi/5$ turn. The pentagon on the plane 2 is next to the top, and so on. The edges are labeled as 12, 23, ...; for example, 12 indicates the intersection of

two pentagons on the planes 1 and 2. The edges labeled by two consecutive numbers are the special edges. The special edges form a circle:

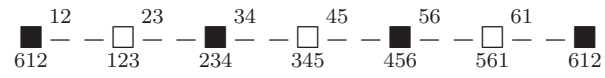
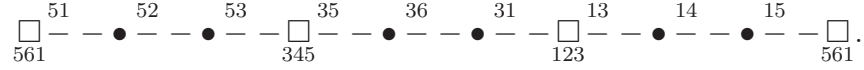


Figure 18: Piles of six dumplings glued along their pentagonal faces (pileofG1)

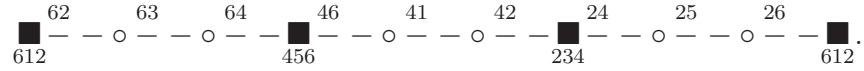
One can see in Figure 18, in front, the quadrilateral consisting of two triangles and two rectangles with edges labeled 12 at the top and 61 in the bottom, and segments 31, 41 and 51 are inside; this rectangle together with the pentagon 12 – 13 – 14 – 15 – 61 is on the plane 1.

3.4.3 Some curves on the dumpling torus

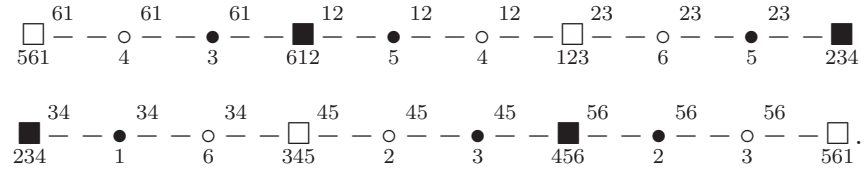
Trace the edge 51 on this rectangle along non-special edges, and we have the close curve:



Trace the edge 41 on this rectangle along non-special edges, and we have the close curve:



The other edges form a single closed curve:



Boundary of a(ny) pentagon; they will be called m_D later:



The diagonal of the front quadrilateral in Figure 18 connecting \blacksquare_{612} crossing the edges 31, 41 and 51; this will be called L later.

3.4.4 Another description of the dumpling torus

The dumpling torus presented as the pile of dumplings above can be understood also as follows, which may help the understanding: Consider the cylinder made by six copies of a pentagon times the interval $[0, 1]$, glued along pentagonal faces; the top and the bottom with $2\pi/5$ twist. The side of the cylinder is tessellated by 5×6 rectangular tiles. Now choose six squares diagonally arranged (thanks to the twist, one can make it consistently) and compress these squares vertically, then you end up with the pile of dumplings above. In Figure 19, it is shown in particular, how the dumpling with pentagonal faces 5 and 6 is made by a pentagon times the interval.

Figure 19: Compressing the pile of six pentagonal prisms to the dumpling torus (chochin)

3.4.5 The boundary of the dumpling torus and the line L

The boundary of the dumpling torus on each plane is shown in Figures 5 and 4, and more explicitly again in Figure 20. It consists of two quadrilaterals DP and two triangles DT . This is the front rectangle shown in Figure 18, and a column of Figure 19 left. We show it again below by a diagram which will fit the diagram of the boundary of the prism torus shown in §3.3.4.



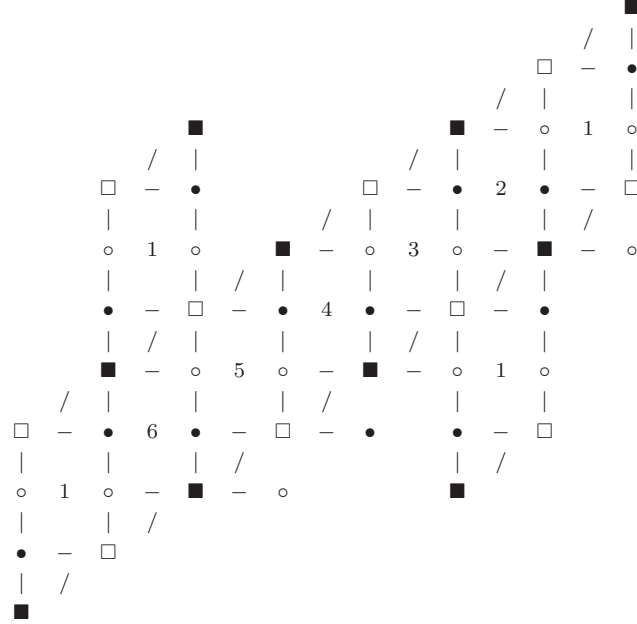
Note that in the left diagram, the two opposite \blacksquare stands for the same vertex; in the right diagram, two opposite \square stands for the same vertex. The diagonal joining two opposite \blacksquare in the left (in Figure 20 left, the line $x - y + 2 = 0$) and the diagonal joining two opposite \square in the right (in Figure 20 right, a vertical line $y = 1/2$) are closed curves, which are homotopically equivalent on the boundary of the dumpling torus; Let us call one of them L .

Figure 20: Boundary of the dumpling torus on each plane (planeH4-70)

We now show the six such forming the boundary of the dumpling torus. The twelve quadrilaterals and the twelve triangles tessellate the boundary torus as below. To indicate the identification, four copies of the plane number 1 are shown. After identification, there are

$$3 \square, \quad 3 \blacksquare, \quad 6 \bullet, \quad 6 \circ;$$

note that among the twenty vertices, only P^0 and P_0 are missing.



3.5 Tetrahedra

Let us compare the (tessellated) boundary of the prism torus and that of the dumping torus. They share the quadrilateral faces. The two solid tori almost fill the space; six tetrahedra are the notches. Each tetrahedron has two triangular faces adjacent to prisms and other two triangular faces adjacent to dumpings:

$$\partial \begin{array}{c} \circ - \blacksquare \\ | \times | \\ \square - \bullet \end{array} = \begin{array}{c} \circ - \blacksquare \\ | \setminus | \\ \square - \bullet \end{array} \cup \begin{array}{c} \circ - \blacksquare \\ | / | \\ \square - \bullet \end{array}$$

In Figures 8(right) and 9(left), we see that each tetrahedron has bounded base (triangle $(\bullet, \circ, \blacksquare) \subset H^\circ$) on the big-cylinder, an infinitely long face on the prism torus, and two infinitely long faces on the dumping torus; intersection of the last two faces is a special edge. The six tetrahedra form a rosary (see Figure 21(left) for a combinatorial idea).

The curve \check{K} , introduced in §3.2.3 lives in the union of the six tetrahedra.

3.6 Intersection with a big sphere

The aim of this section is to visualize what happens near the plane at infinity H_∞ by looking at the intersection with a big sphere.

The sphere at infinity is shown in Figure 22; the plane at infinity (see Figure 6) is obtained by identifying antipodal points. The three circles represent (the intersections with the planes) H_x, H_y, H_z , showing an octahedron, and the three lines represent (the intersections with the planes) H^x, H^y, H^z . They cut out 24 triangles. Each triangle is a section of a prism or a dumping. Dotted circles represent (the intersection with) the surface $Q = 0$.

A very big sphere is shown in Figure 23. The three circles (showing an octahedron) and the three lines represent as in the previous figure. The triangle appeared at the center is the section of a cube; the other cube situates out of the picture. The six small triangles are the sections of the tetrahedra.

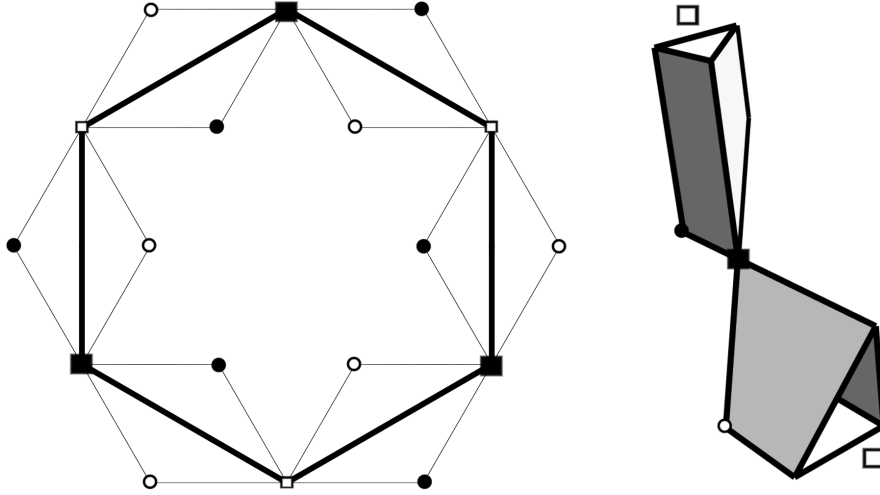


Figure 21: The rosary of tetrahedra and two tetrahedra extracted from Figure 8(right)

Figure 22: Intersection of the chambers with the sphere at infinity (sphere0)

Each prism P is the union of two parts P' and P'' as is shown in Figure 7, glued along the shaded triangle. Now the shaded triangle is thickened; the sections of P' and P'' with the big sphere are a triangle and a pentagon, respectively. (See Figure 24 left.)

Each dumpling D is the union of two parts D' and D'' as is shown in Figure 9, glued along the shaded triangle. Now the shaded triangle is thickened; the sections of D' and D'' with the big sphere are both quadrangle. (See Figure 24 right.)

Intersection with the big sphere with the surface $Q = 0$ is shown in Figure 25; which shows that the surface does not path through any prisms nor cubes, and that the surface can be considered as an approximation of (the boundary of) the prism torus.

Figure 23: Intersection of the chambers with a big sphere (sphere1)

Figure 24: Enlargement of two faces of the octahedron (sphere3)

Figure 25: Enlargement around the intersection with a tetrahedron together with that of the surface $Q = 0$ (sphere2)

4 Seven hyperplanes in the 4-space

Seven hyperplanes in general position in \mathbb{P}^4 cut out a unique chamber, bounded by the seven hyperplanes, stable under the action of the cyclic group \mathbb{Z}_7 ; let us call it the central chamber CC. This action can be projective if the hyperplanes are well-arranged. The intersection of CC and a hyperplane is a dumpling.

So the boundary of CC is a 3-sphere \mathbb{S}^3 tessellated by seven dumplings. We would like to know how they are arranged, especially the \mathbb{Z}_7 -action on the tessellation. In [CY], the seven dumplings in \mathbb{S}^3 is shown (see Figure ??), in which the \mathbb{Z}_7 -action can be hardly seen.

We first observe the tessellation. We label the dumplings as D_1, D_2, \dots, D_7 so that the dumplings D_k and D_{k+1} , modulo 7, share a pentagon, and the group action induces a transformation $D_k \rightarrow D_{k+1}$ modulo 7. The special edges form a circle C . (Recall that a dumpling is bounded by two pentagons, two triangles and two quadrilaterals, and that the intersection of the two pentagons is called the special edge.) If we remove seven pentagons $D_k \cap D_{k+1}$ from the 2-skeleton of the tessellation, it remains a Möbius strip M with the curve C as the boundary.

We find, in the 2-skeleton, a subskeleton homeomorphic to a disc which bound the curve C . This implies that C is unknotted, and that M is twisted only by $\pm\pi$.

We give another description of the tessellation so that the action of \mathbb{Z}_7 can be seen; we start from a vertical pentagonal prism, horizontally sliced into seven thinner prisms. We collapse vertically the seven rectangular faces diagonally situated on the boundary of the prism; this process changes each thinner prism into a dumpling. Identifying the top and the bottom after a suitable rotation, we get a solid torus ST made of seven dumplings, so that the solid torus ST admits a \mathbb{Z}_7 -action. Consider a usual solid torus UT in our space \mathbb{S}^3 and put ST outside of UT ; the boundary of ST , which is also the boundary of UT , is a torus T . We collapse UT by folding the torus T into a Möbius strip, which can be identified with M , whose boundary can be identified with C . We will see that the result, say X_7 , of this collapsing is homeomorphic to \mathbb{S}^3 . In this way, we recover the tessellation of \mathbb{S}^3 by seven dumplings.

To make our idea and a possible inductive process clear, we start this chapter by studying a chamber bounded by five (as well as six) hyperplanes stable under the action of the cyclic group \mathbb{Z}_5 (resp. \mathbb{Z}_6); though such a chamber is not unique, the above statement with obvious modification is still true, if we understand the special edge of D_i as $D_{i-1} \cap D_i \cap D_{i+1}$. A 3-dumpling is a tetrahedron, and a 4-dumpling is a prism. After the collapsing process as above, we get the manifolds X_5 (resp. X_6). We prove in detail that X_5 is homeomorphic to \mathbb{S}^3 ; for other cases proof is similar.

4.1 Five hyperplanes in the 4-space

Five hyperplanes in the projective 4-space cut out sixteen 4-simplices. If a hyperplane is at infinity, then the remaining four can be considered as the four coordinate hyperplanes, dividing the space into 2^4 chambers. The boundary of each 4-simplex is of course made by five tetrahedra. In other words, five tetrahedra are glued together to make a (topological) 3-sphere: around a tetrahedron, glue four tetrahedra along their faces, and further use the valley of each pair of triangles in order to close the pair like a book.

4.1.1 Labeling and a study of 2-skeleton

We study the above tessellation more precisely. Label the five hyperplanes as H_1, \dots, H_5 . Choose one 4-simplex, out of sixteen, and call it CC. We use the convention

$$H_{ij} = H_i \cap H_j, \quad H_{ijk} = H_i \cap H_j \cap H_k, \dots$$

The five vertices of CC are:

$$H_{2345}, \quad H_{1345}, \quad \dots, \quad H_{1234}$$

which are often denoted simply by

$$1, 2, \dots, 5,$$

respectively. The boundary of CC is made by five tetrahedra:

$$D_1 = 2345, \quad D_2 = 1345, \quad \dots$$

(Tetrahedron D_1 has vertices 2, 3, 4, 5.) Any two of them share a triangle:

$$D_i \cap D_j = \Delta(\{1, \dots, 5\} - \{i, j\}), \quad i \neq j;$$

there are ten of them. These tetrahedra tessellate the 3-sphere \mathbb{S}^3 . We consider the sequence D_1, D_2, \dots , in this order, and name the intersections of the two consecutive ones as:

$$D_1 \cap D_2 = \Delta(345), \quad D_2 \cap D_3 = \Delta(451), \quad \dots \quad D_5 \cap D_1 = \Delta(234).$$

(Note that the union of these five triangles form a Möbius strip, but for a while just forget it.) Remove these five triangles from the 2-skeleton of the tessellation. Then five triangles remain, forming another Möbius strip

$$M : \quad \begin{array}{ccccccc} 3 & - & 4 & - & 5 & - & 1 \\ & \backslash & / & \backslash & / & \backslash & / \\ & 1 & - & 2 & - & 3 & \end{array} .$$

Its boundary

$$C = \partial M : \quad 1 - 2 - 3 - 4 - 5 - 1$$

is an unknotted circle because C bounds also the disc

$$\Delta(234) \cup \Delta(124) \cup \Delta(451) : \quad \begin{array}{ccccccc} 3 & - & 4 & - & 5 \\ & \backslash & / & \backslash & / \\ & 2 & - & 1 & \end{array} .$$

4.1.2 Visualization

Since the situation is quite simple, we visualize the happenings. Consider a tetrahedron with vertices $\{1, 2, 3, 4\}$ in our space, and put vertex 5 at the barycenter. The barycentric subdivision of this tetrahedron (which yields D_1, \dots, D_4) together with its complementary tetrahedron D_5 fills the 3-sphere. Figure 26 shows the 2-skeleton consisting of ten triangles as the union of the Möbius strips M and the complementary one. You can find the disc, given in the previous subsection, bounded by the curve C . When you make a paper model of the 2-skelton out of Figure 26, be ware that the two Möbius strips have different orientations.

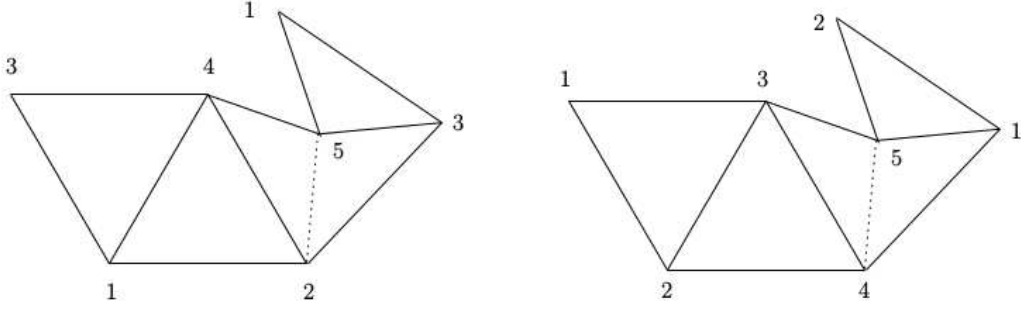


Figure 26: Möbius strips M (left) and the complementary one (right) before identification along appropriate edges.

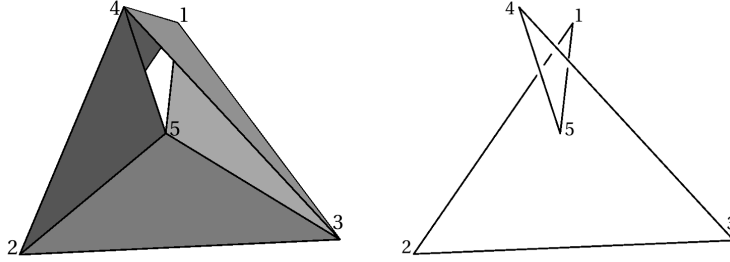


Figure 27: (left) Möbius strip M . (right) Boundary ∂M .

4.1.3 From a solid torus

We would like to give another description, which will prepare the next step where the number m of hyperplanes is $m = 6, 7, \dots$, in such cases, naive description above would hardly work. We start with a *vertical triangular prism*, horizontally sliced into five thinner triangular prisms. Its top and the bottom triangles are identified after a $4\pi/3$ rotation; this makes the vertical prism a solid torus. The boundary torus is tessellated by fifteen rectangles. Now collapse vertically five diagonal rectangles. This changes every remaining rectangles into triangles, and the thinner triangular prisms into tetrahedra. Further, use each collapsed edge as a valley of each pair of triangles face-to-face in order to close the pair like a book. This identification makes the solid torus a 3-sphere tessellated by five tetrahedra, recovering the tessellation described in §4.1.1.

A vertical triangular prism horizontally sliced. We show/repeat the above process by using the labels 1, 2, 3, 4, 5 (they correspond the labels of the five hyperplanes H_1, H_2, \dots). Consider five triangles labeled by two consecutive numbers: 51, 12, ... (they correspond the triangle $H_{51} \cap CC, \dots$). The triangle 51, for example, has three vertices with labels

$$51 : \{5124, 5123, 5134\}$$

(they correspond the points $\{H_{4512}, H_{5123}, H_{3451}\}$). Now operate the group \mathbb{Z}_5 on the labels: $j \rightarrow j + 1 \pmod{5}$, and we get four other triangles, for example, the triangle 12 has three vertices:

$$12 : \{1235, 1234, 1245\}.$$

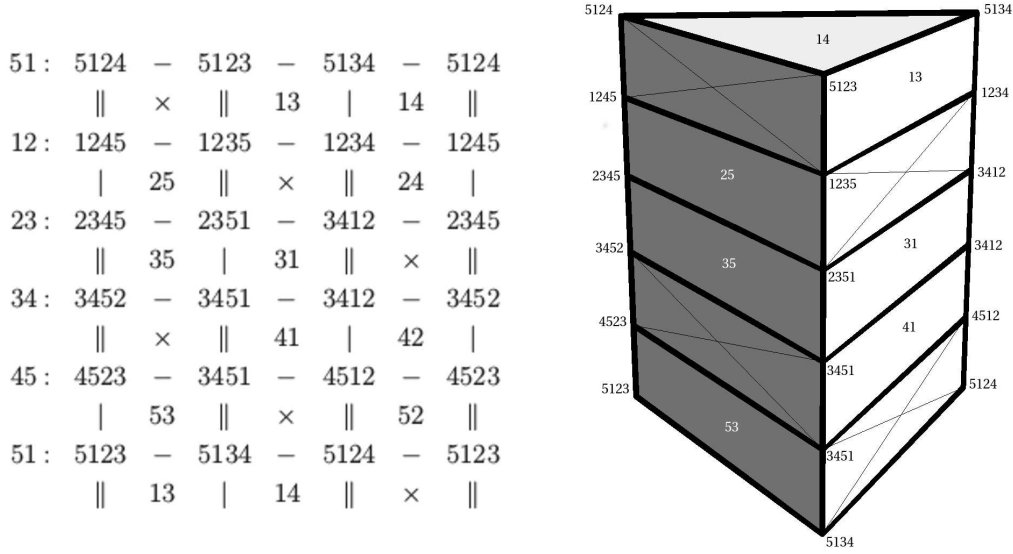


Figure 28: (left) Table of five piles. (right) Triangular prism split up into five slices. The label 14 denotes the upper back rectangle.

We make a triangular prism by putting the triangle 51 above the triangle 12; the two vertices of 51 having the same labels of the ones of 12 should be put just above each other. In a word this prism is the sandwich made by the two triangles 51 and 12.

In this way, we make five sandwiches and pile them to form a vertical triangular prism. The table in Figure 28 repeats what we described. Each off-diagonal rectangle is labeled by the two numbers common to the four vertices.

Collapsing rectangles. If we identify two points with the same indices *as sets*, for example, $4512 = 5124 = 1245$, you find five rectangles with pairwise coincide vertices (marked by \times), situated diagonally. We collapse such rectangles vertically to a segment, each called a **special edge**. Note that a special edge has two ends which are labeled by consecutive numbers: $\{1234, 2345\}, \{2345, 3451\}, \dots$. Accordingly all the remaining rectangles become triangles and, the five thin prisms become tetrahedra. But as a whole it still remains to be a cylinder. This collapsing process is described in Figure 29. Here the prisms are hollow so that we can peep inside.

Identifying the top and the bottom to get a solid torus ST . We identify the top triangle and the bottom triangle of the collapsed vertical prism according to the labels; note that we must twist the triangle by $4\pi/3$. By the identification we get a solid torus, say ST ; its meridean **mer** is represented by the boundary of a(ny) triangle, in Figure 28, say 51 (or 12, 23, ...). Note that it is still an abstract object. The five special edges now form a circle:

$$C : 1234 - 2345 - 3451 - 4512 - 5123 - 1234.$$

We take a vertical line, in Figure 28 (left), with a $4\pi/3$ twist, say,

$$\text{par} : 5123 - 3451 - 5123 = 4 - 2 - 4$$

as a parallel.

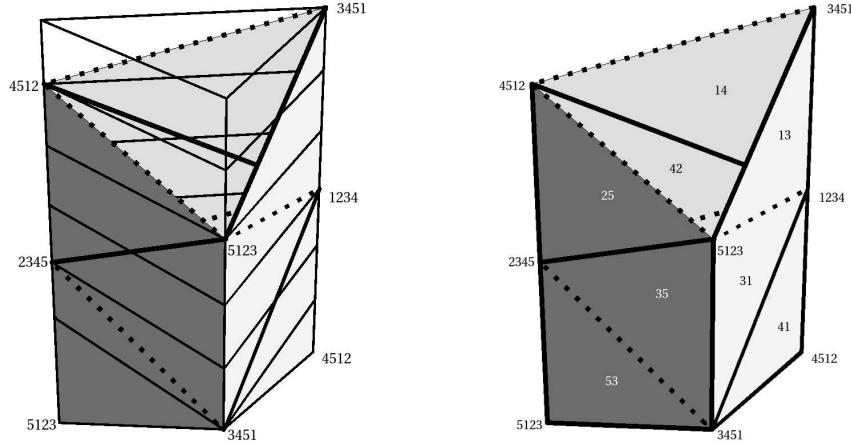


Figure 29: (left) Collapsing rectangles. (right) Tiling the boundary of the solid cylinder by ten triangles. The dashed line is the special loop C .

The torus T and the solid torus UT . We consider in our space ($\sim \mathbb{S}^3$) a usual (unknotted) solid torus, say UT , and put our solid torus ST fills outside of UT , so that the tessellated boundary torus

$$T := \partial ST (= \partial UT)$$

has **par** as the meridean, and **mer** as the parallel. From the description of the vertical prism, we see ten triangles tessellating the torus T arranged in a hexagonal way (Figure 30-left).

Note that the triangulated torus above is not a simplicial complex; indeed there are two edges with the same vertices, for example, $5123 - 3451$. Anyway, you can see the curve C lies on the torus $T = \partial ST$ as a $(2, 1)$ -curve (see Figure 31).

Collapsing the solid torus UT by folding T . On the torus T , there are pairs of triangles with the same vertices. We identify these triangles by folding the torus T along the curve C . This collapses the solid torus UT ; we prove below that the resulting space, say X_5 , is still \mathbb{S}^3 . The torus T is folded to be a Möbius strip M with boundary C :

$$M : \begin{array}{ccccccc} 4512 & - & 5123 & - & 1234 & - & 2345 \\ & \backslash & 25 & / & 35 & \backslash & 31 & / & 41 & \backslash & 42 & / \\ & & 2345 & - & 3451 & - & 4512 \end{array}$$

Note that M is a simplicial complex. If we use complementary labeling, for example, 5 for 1234, this is exactly the same as we got several pages before. On the hexagonal expression of the torus T , this folding is done as closing a book along the diagonal.

The process of collapsing the solid torus UT onto the Möbius strip M can be explicated by the following homotopy parametrized by $t \in [0, 1]$:

$$F(\vartheta, \eta, r, t) = O' + (1 - t)r\vec{w}(\vartheta, \eta + \frac{\vartheta}{2}) + tr \cos \eta \cdot \vec{w}(\vartheta, \frac{\vartheta}{2}),$$

where

$$O' = \sqrt{2}(\cos \vartheta, \sin \vartheta, 0), \quad \vec{w}(\vartheta, \eta) = (\cos \eta \cos \vartheta, \cos \eta \sin \vartheta, \sin \eta)$$

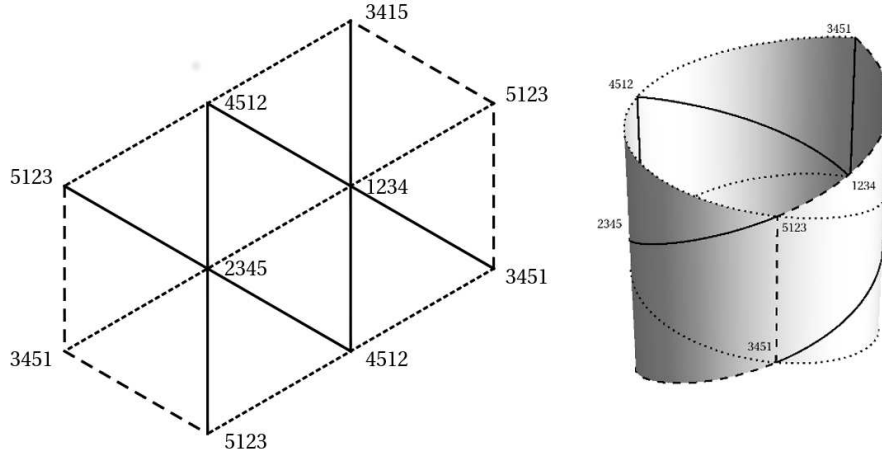


Figure 30: (left) Ten triangles tessellating the hexagon. (right) Identifying the left and right vertical sides we get a cylinder topologically equivalent to that of Figure 29-right. The special curve C is dotted while ST -parallel is dashed.

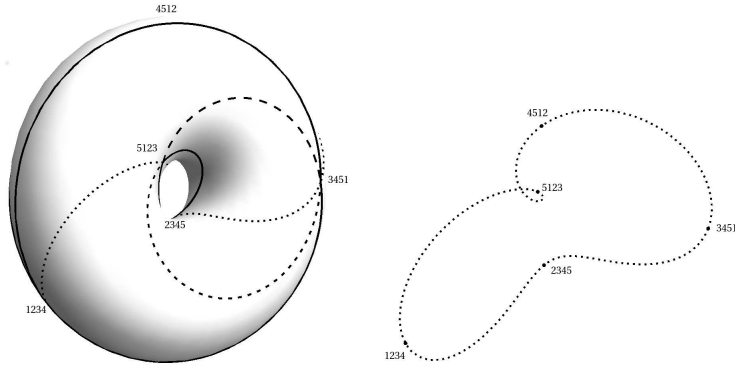


Figure 31: (left) Tessellation of the torus ∂ST . Special curve C is dotted while ST -parallel is dashed. (right) The special curve C .

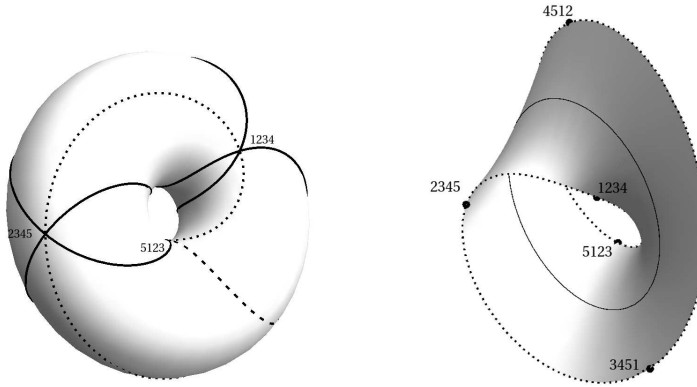


Figure 32: (left) Tessellation of the torus ∂UT . Special curve C is dotted while UT -meridian is dashed. (right) Möbius strip M .

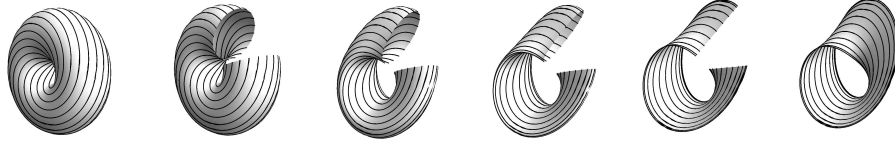


Figure 33: (from left to right) Folding the torus T along the special curve C to be the Möbius strip M . A sector is cut out to show the folding.

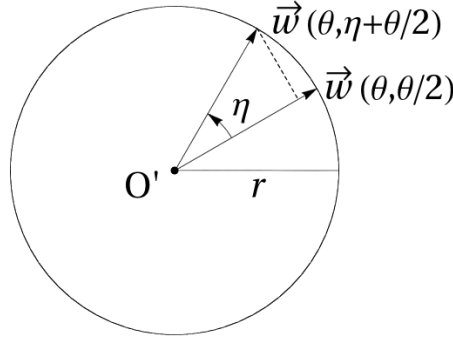


Figure 34: Meridian section of UT . The homotopy collapsing UT onto M follows the dashed line.

and $(\exp(i\vartheta), \exp(i\eta), r)$ parametrizes the solid torus $\mathbb{S}^1 \times \mathbb{D}^2$ (Figure 34).

Puffing the Möbius strip to make a torus. The inverse operation of folding the (unknotted) torus T to the Möbius strip M (with $\pm\pi$ twist) can be described as follows: Consider a Möbius strip M being double sheeted like a flat tire folded along C . Then puff the tire to be a air-filled tire T . It is illustrated in Figure 33 seen from right to left.

A disc bounding C . Let us repeat the disc in §4.1.1 bounding C , using complementary labeling:

$$\begin{array}{ccccccc}
 4512 & - & 5123 & - & 1234 & & \\
 \backslash & 51 & / & 35 & \backslash & 23 & / \\
 3451 & - & 2345 & & & &
 \end{array}$$

Proof that X_5 is homeomorphic to \mathbb{S}^3 . The collapsing of $\mathbb{S}^3 = UT \cup_T ST$ by folding T along C and collapsing UT on M is nothing but the gluing of the set of the five tetrahedra tiling ST in such a way that the tetrahedra facets are identified by pairs. In doing so we must get ST back in one hand, and, on the other hand, the triangles of the boundary ∂ST are identified by pairs to get the Möbius strip M . The result X_5 , is a closed 3-dimensional manifold since the link of each vertex is homeomorphic to a 2-sphere, namely the 2-skeleton of a tetrahedron. For instance the link of the vertex 1234 is the union of the four triangles $\{15, 25, 35, 45\}$, where 15 for example, is the triangle with vertices 1523, 1534 and 1542. The 3-manifold X_5 , as a continuous image of \mathbb{S}^3 , is clearly connected. Now, computing the fundamental group $\pi(X_5, *)$ will prove that X_5 is simply connected and therefore, by Poincaré-Perelman theorem, homeomorphic to \mathbb{S}^3 . Indeed, since X_5 is endowed with a structure of simplicial complex, $\pi(X_5, *)$ is isomorphic to $\pi(X_5^2, *)$, where X_5^2 denotes the 2-skeleton. The group $\pi(X_5^2, *)$ can be described by generators and

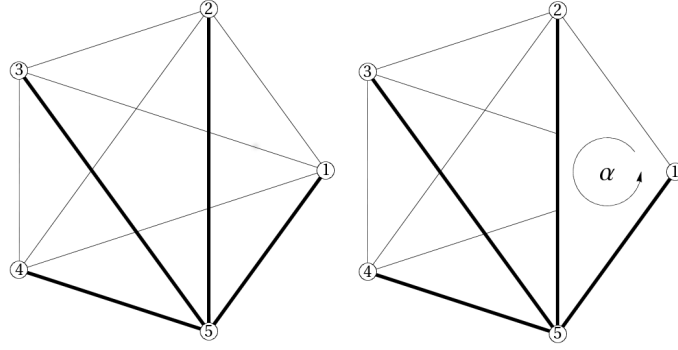


Figure 35: (left) The complete graph X_5^1 over five vertices and a maximal tree base at 5. (right) One of the six generators of $\pi(X_5^1, 5)$: $\alpha = (5, 1, 2)$.

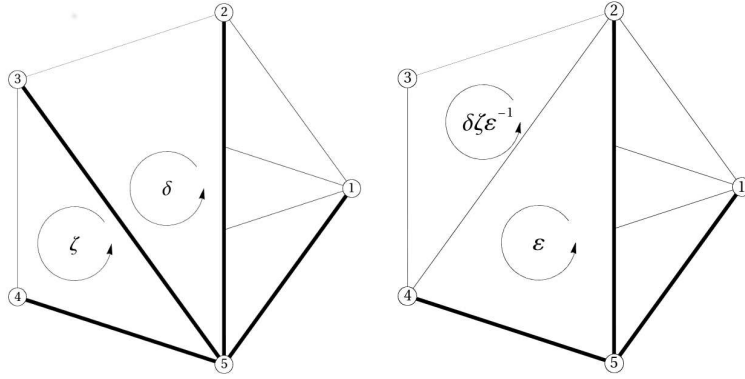


Figure 36: (left) The generators $\delta = (5, 2, 3)$ and $\zeta = (5, 3, 4)$. (right) The generator $\varepsilon = (5, 2, 4)$ and the relator $\delta\zeta\varepsilon^{-1} = (2, 3, 4)$.

relators as follows. Firstly, we observe that the 1-skeleton X_5^1 is the complete graph over the five vertices 1, 2, 3, 4, 5 (using complementary labeling) (Figure 35).

Recall that we can use a maximal tree (that is a tree containing all vertices) to compute the fundamental group $\pi(X_5^1, 5)$. Indeed, to each edge not contained in the tree, we associate in an obvious way a loop based at 5. The set of these loops is a basis of the free group $\pi(X_5^1, 5)$. Therefore we see that $\pi(X_5^1, *)$ is the free group over the six following generators (still using complementary labeling for the sequence of vertices and choosing 5 as a base point):

$$\alpha = (5, 1, 2), \quad \beta = (5, 1, 3), \quad \gamma = (5, 1, 4), \quad \delta = (5, 2, 3), \quad \varepsilon = (5, 2, 4), \quad \zeta = (5, 3, 4)$$

Secondly, the relators of $\pi(X_5^2, *)$ are associated with the ten facets coming from the five tetrahedra tiling X_5 :

$$12, \quad 13, \quad 14, \quad 15, \quad 23, \quad 24, \quad 25, \quad 34, \quad 35, \quad 45.$$

It yields to the following relators (Figure 36):

$$\zeta, \quad \varepsilon, \quad \delta, \quad \delta\zeta\varepsilon^{-1}, \quad \gamma, \quad \beta, \quad \beta\zeta\gamma^{-1}, \quad \alpha, \quad \alpha\varepsilon\gamma^{-1}, \quad \alpha\delta\beta^{-1},$$

something which proves that

$$\begin{aligned} \pi(X_5, *) &= \pi(X_5^2, *) = \\ &< \alpha, \beta, \gamma, \delta, \varepsilon, \zeta : \zeta = \varepsilon = \delta = \delta\zeta\varepsilon^{-1} = \gamma = \beta = \beta\zeta\gamma^{-1} = \alpha = \alpha\varepsilon\gamma^{-1} = \alpha\delta\beta^{-1} = 1 > \\ &= 1. \end{aligned}$$

Conclusion. In the sphere $\mathbb{S}^3 = UT \cup_T ST$, the solid torus UT is collapsed to an unknotted Möbius strip M , which is twisted only by $\pm\pi$ (see Figure 33), and the resulting 3-manifold X_5 is a 3-sphere. In this way we recover the tessellation of $\partial\Delta_4 = \partial CC$ described in §4.1.1.

4.2 Six hyperplanes in the 4-space

Six hyperplanes in the projective 4-space cut out six 4-simplices, fifteen prisms of type $\Delta_3 \times \Delta_1$ and ten prisms of type $\Delta_2 \times \Delta_2$. If a hyperplane is at infinity, then the remaining five bound a simplex. Other chambers touch this simplex along 3-simplices (five $\Delta_3 \times \Delta_1$), along 2-simplices (ten $\Delta_2 \times \Delta_2$), along 1-simplices (ten $\Delta_3 \times \Delta_1$) and along 0-simplex (five Δ_4).

Though a prism of type $\Delta_3 \times \Delta_1$ is bounded by six hyperplanes, it does not admit an action of the group \mathbb{Z}_6 . So we consider one of the prisms of type $\Delta_2 \times \Delta_2$, which is bounded by six prisms (of type $\Delta_2 \times \Delta_1$). We study how these six prisms tessellate a 3-sphere.

4.2.1 Labeling and a study of 2-skeleton

Let the six hyperplanes H_1, \dots, H_6 bound a chamber CC of type $\Delta_2 \times \Delta_2$. The chamber CC has six faces D_1, \dots, D_6 , which form the \mathbb{Z}_6 -orbit of the prism

$$D_1 = CC \cap H_1 : \begin{array}{ccccc} 1346 & -- & -- & -- & 1456 \\ | & \backslash & 4 & / & | \\ | & 3 & 1234 & -- & 1245 & 5 & | \\ | & / & 2 & \backslash & | \\ 1236 & -- & -- & -- & 1256 \end{array},$$

where $1234 = H_{1234} = H_1 \cap \dots \cap H_4$. Note that the rectangular face of D_1 behind is $D_1 \cap D_6$, and the rectangular face in front below is $D_1 \cap D_2$.¹ The 2-skeleton of CC consists of six triangles and nine rectangles. Remove the intersections of the two consecutive ones:

$$D_1 \cap D_2, \quad D_2 \cap D_3, \quad \dots, \quad D_6 \cap D_1 :$$

¹Using the notation introduced in Chapter 2, CC is represented by $+++++$, and its boundary consists of

$$D_1 = 0++++, \quad D_2 = -0++++, \quad \dots, \quad D_6 = ----0.$$

The boundary of D_1 consists of (cf. §??)

$$\begin{array}{lll} 00++++ : 4\text{-gon} & 0-0+++ : \text{triangle} & 0--0++ : \text{rectangle} \\ 0---0+ : \text{triangle} & 0---0 : 4\text{-gon} & \end{array}$$

$$\begin{array}{ccccccccc}
5612 & - & 6123 & - & 2356 & & & & \\
| & 12 & | & 23 & | & & & & \\
1245 & - & 1234 & - & 2345 & - & 4512 & & \\
& & | & 34 & | & 45 & | & & \\
& & 3461 & - & 3456 & - & 4561 & - & 6134 \\
& & & & | & 56 & | & 61 & | \\
& & & & 5623 & - & 5612 & - & 6123 .
\end{array}$$

(Note that the union of these six rectangles form a cylinder, *not* a Möbius strip.) Then the remaining six triangles and three rectangles form a Möbius strip

$$\begin{array}{ccccccccccc}
5612 & - & 6123 & - & 1234 & - & 2345 & - & 3456 \\
& \backslash & 26 & / & \backslash & 31 & / & \backslash & 42 & / & \backslash & 53 & / \\
M : & & 2356 & & & 3461 & & & 1245 & & & 2356 \\
& & \backslash & 36 & / & 46 & \backslash & 41 & / & 51 & \backslash & 52 & / \\
& & & 5634 & - & & 6145 & - & & 5612 & & &
\end{array}$$

Möbius 6

Its boundary

$$C = \partial M : \quad 1234 - 2345 - 3456 - 4561 - 5612 - 6123 - 1234$$

is an unknotted circle because C bounds also the disc

$$\begin{array}{ccccccc}
6123 & & - & & 1234 & & \\
/ & & & \backslash & 13 & / & \backslash \\
5612 & 61 & & 6134 & 34 & 2345 & . \\
& \backslash & & / & 46 & \backslash & / \\
4561 & & - & & 3456 & &
\end{array}$$

4.2.2 From a solid torus

We start with a vertical rectangular prism, horizontally sliced into six thinner rectangular prisms. Its top and the bottom rectangles are identified after a $4\pi/4$ rotation; this makes the vertical prism a solid torus. The boundary torus is tessellated by 24 rectangles. Now collapse vertically six diagonal rectangles. This changes eighteen rectangles (out of 24) into six segments (forming a circle) and twelve triangles. Accordingly, the six thinner rectangular prisms (actually cubes) into prisms. Next use each collapsed edge as a valley of each pair of triangles face-to-face in order to close the pair like a book, and further identify three pairs of rectangles. This identification changes the solid torus into a 3-sphere tessellated by six prisms.

A vertical rectangular prism horizontally sliced. We show/repeat the above process by using the labels 1, 2, 3, 4, 5, 6 (they correspond the labels of the six hyperplanes H_1, H_2, \dots). Consider six rectangles labeled by two consecutive numbers: 61, 12, ... (they correspond the rectangle $H_{61} \cap CC, \dots$). The rectangle 61, for example, has four vertices with labels

$$61 : \quad \{6125, 6123, 6134, 6145\}$$

(they correspond the points $\{H_{5612}, H_{6123}, H_{6134}, H_{6145}\}$). Now operate the group \mathbb{Z}_6 on the labels: $j \rightarrow j + 1 \bmod 6$, and we get five other rectangles, for example, the rectangle 12 has three vertices:

$$12 : \quad \{1236, 1234, 1245, 1256\}.$$

We make a rectangular prism by putting the rectangle 61 above the rectangle 12; the two vertices of 61 having the same labels of the ones of 12 should be put just above each other. In a word this prism is the sandwich made by the two rectangles 61 and 12.

In this way, we make six sandwiches and pile them to form a vertical rectangular prism. The following table repeats what we described. Each off-diagonal rectangle is labeled by the two numbers common to the four vertices.

61 :	6125	—	6123	—	6134	—	6145	—	6125
		×		13		14		15	
12 :	1256	—	1236	—	1234	—	1245	—	1256
		26		×		24		25	
23 :	2356	—	2361	—	2341	—	2345	—	2356
		36		31		×		35	
34 :	3456	—	3461	—	3412	—	3452	—	3456
		46		41		42		×	
45 :	4563	—	4561	—	4512	—	4523	—	4563
		×		51		52		53	
56 :	5634	—	5614	—	5612	—	5623	—	5634
		64		×		62		63	
61 :	6134	—	6145	—	6125	—	6123	—	6134
		14		15		×		13	

Table of piles 6

Collapsing rectangles. If we identify two points with the same *consecutive* indices *as sets*, for exmaple, $6123 = 1236 = 2361$ but $2356 \neq 5623$, you find six rectangles with pairwise coincide vertices (marked by \times), situated diagonally. We collapse such rectangles vertically to a segment, each called a **special edge**. Note that a special edge has two ends which are labeled by consecutive numbers: $\{1234, 2345\}, \{2345, 3451\}, \dots$ Accordingly the remaining rectangles next to the special edges become triangles, and the six thin rectangular prisms become triangular prisms ($\sim \Delta_2 \times \Delta_1$). But as a whole it still remains to be a cylinder.

Identifying the top and the bottom to get a solid torus ST . We identify the top rectangle and the bottom rectangle of the collapsed vertical prism according to the labels; note that we must twist the triangle by $4\pi/4$. By the identification we get a solid torus, say ST ; its meridean **mer** is represented by the boundary of a(ny) rectangle, in Table of piles 6, say 61 (or 12, 23, ...). Note that it is still an abstract object. The six special edges now form a circle:

$$C : 1234 - 2345 - 3456 - 4561 - 5612 - 6123 - 1234.$$

We take a vertical line, in Table of piles 6, with a $4\pi/4$ twist, say,

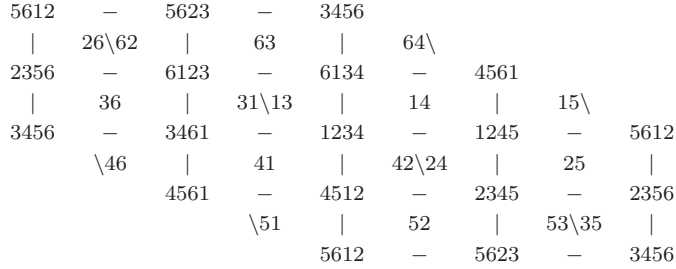
$$\mathbf{par} : 6123 - 3461 - 4561 - 6123$$

as a parallel.

The torus T and the solid torus UT . We consider in our space ($\sim \mathbb{S}^3$) a usual (unknotted) solid torus, say UT , and think our solid torus ST fills outside of UT , so that the tessellated boundary torus

$$T := \partial ST (= \partial UT)$$

has **par** as the meridean, and **mer** as the parallel. From the description of the vertical prism, we see twelve triangles and six rectangles tessellating the torus T arranged in a hexagonal way (identify the opposite sides of the hexagon):



Torus (hexagon) 6

Collapsing the solid torus UT by folding T . We identify two vertices with the same indices as sets. Along each collapsed edge (special edge), there are two triangles with the same vertices; these triangles are folded along the special edge into one triangle. Further there are three pairs of rectangles with the same vertices; they are also identified. Consequently the torus T is folded along the curve C to be a Möbius strip, which turns out to be the same as the one in the previous subsection (shown as Möbius 6). In this way, we get a space X_6 from the solid torus ST made by six prisms by folding the torus T (collapsing UT).

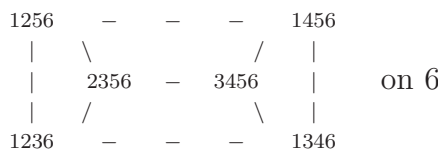
Since the curve C is unknotted, this folding is done exactly the same as in the previous section. Then X_6 is homeomorphic to \mathbb{S}^3 .

A model. Put three vertices with labels without numeral 6: 1245, 2345, 1234 inside the prism (6123, 5612, 6235) * (6134, 6145, 6345). The Möbius strip above is shown in Figure 37 ([CYY]). From this picture, one can hardly see the \mathbb{Z}_6 action.

Figure 37: Five prisms pack a prism. Möbius strip made by six triangles and three rectangles is shown(Moe6)

4.2.3 $6 - 1 = 5$

If we remove the sixth hyperplane, then the remaining five hyperplanes bounds a 4-simplex. This process can be described as follows: the prism on the sixth plane reduces to a segment. More precisely, the two triangles of the boundary of the sixth prism reduce to two points, and the prism reduces to a segment connecting these two points. Combinatorial explanation: The prism on the sixth plane with vertices



reduces $(1256, 2356, 1236 \rightarrow 1235, 1456, 3456, 1346 \rightarrow 1345)$ to the segment

$$\mathbf{1235} - - - \mathbf{1345}.$$

Note that

$$\{1, 2, 3, 5\} = \{1, 2, 5, 6\} \cup \{2, 3, 5, 6\} \cup \{1, 2, 3, 6\} - \{6\},$$

$$\{1, 3, 4, 5\} = \{1, 4, 5, 6\} \cup \{3, 4, 5, 6\} \cup \{1, 3, 4, 6\} - \{6\}.$$

Accordingly the other prisms reduce to tetrahedra, for example:

$$\begin{array}{ccc} \begin{array}{c} 3412 \\ | \quad \backslash \quad / \\ | \quad 4512 - 5612 \\ | \quad / \quad \backslash \\ 3452 \quad - - - \quad 3562 \end{array} & \longrightarrow & \begin{array}{c} 3412 \\ | \quad \backslash \quad \backslash \\ | \quad 4512 - \mathbf{1235} \\ | \quad / \quad / \\ 3452 \end{array} \quad \text{on 2 (similar on 4),} \\ \\ \begin{array}{c} 6145 \\ | \quad \backslash \quad / \\ | \quad 1245 - 2345 \\ | \quad / \quad \backslash \\ 6125 \quad - - - \quad 6235 \end{array} & \longrightarrow & \begin{array}{c} \mathbf{1345} \\ / \quad | \quad \backslash \\ 1245 - 2345 \\ \backslash \quad | \quad / \\ \mathbf{1235} \end{array} \quad \text{on 5 (similar on 1 and 3).} \end{array}$$

Remark. Consider an n -gon in the plane bounded by n lines. If you remove the n -th line, then you get an $(n - 1)$ -gon. During this process, the $n - 3$ sides away from the n -th sides do not change. Consider an n -hedron in the 3-space bounded by n planes. If you remove the n -th plane, you get an $(n - 1)$ -hedron. During this process, faces away from the n -th face remain unchanged. In the present case, all the prism-faces change into tetrahedra.

4.2.4 $5 + 1 = 6$

If we add the sixth hyperplane to the five hyperplanes, then among the sixteen 4-simplices Δ_4 , some are untouched, some are divided into Δ_4 and $\Delta_3 \times \Delta_1$ (with section Δ_3), and some are divided into $\Delta_2 \times \Delta_2$ and $\Delta_3 \times \Delta_1$ (with section $\Delta_2 \times \Delta_1$.) We will describe the change of Δ_4 into $\Delta_2 \times \Delta_2$ via that of these boundaries: the 3-sphere tessellated by five tetrahedra into that tessellated by six prisms.

As a preparation, we would like to explain this cutting process by using a 1-dimension less model: we truncate a vertex of a polyhedron in 3-space. Let us consider a polyhedron in the 3-space, say for example a tetrahedron bounded by four faces $\{0, 1, 2, 3\}$, and cut the vertex $123 = 1 \cap 2 \cap 3$ by the new plane named 4. The tetrahedron is truncated and is divided into a prism and a small tetrahedron; they share a triangle which should be called 4. Instead of describing this process by 3-dimensional pictures, we express it by 2-dimensional pictures describing the change near the vertex 123: the plane is divided into three chambers 1, 2, 3 (imagine the letter Y) sharing a unique point representing the vertex 123, which we call the center. The cutting is expressed by a blowing-up at the center, which means inserting a triangle in place of the center point (see Figure 38, first line). Though this is not so honest as 3-dimensional pictures, this is still a fairly honest way to describe the cutting process.

There is a way to make these pictures simpler without losing any information: We watch only one chamber, say 3, and show the cutting/blowing-up of the center only in the

Figure 38: Blowing up a point to be a triangle (blowup2)

chamber 3 (see Figure 38, second line, right). You see that the terminal figure of marked triangle tells everything.

Now we are ready. Label the tetrahedra as $1, 2, \dots, 5$ as in the previous subsection. We choose the edge E joining two vertices 1235 and 1345. Along this edge E , there are three tetrahedra 1, 3 and 5. Other than these three, tetrahedron 2 touches the vertex 1235, and tetrahedron 4 touches the vertex 1345. We cut the edge by a new hyperplane 6. The tetrahedron 2 (resp. 4) has vertex 1235 (resp. 1345) as the intersection with E ; this vertex is cut, and the tetrahedron 2 becomes a prism (see Figure 39). The tetrahedron

Figure 39: Truncating a vertex of a tetrahedron (tetratoprism1)

5 (as well as 1 and 3) has the edge E ; this edge is cut, and the tetrahedron 5 becomes a prism (see Figures 40 and 3). The new face 6 is a prism bounded by two triangles

Figure 40: Truncating an edge of a tetrahedron (tetratoprism2)

labeled 2 and 4, and three rectangles labeled 1, 3 and 5. Recall the above 1-dimension lower process, then the blow-up of the edge E to become the prism 6 can be summerized by Figure 41. In the case of seven or more hyperplanes, we describe the cutting/blow-up process only by this kind of figure.

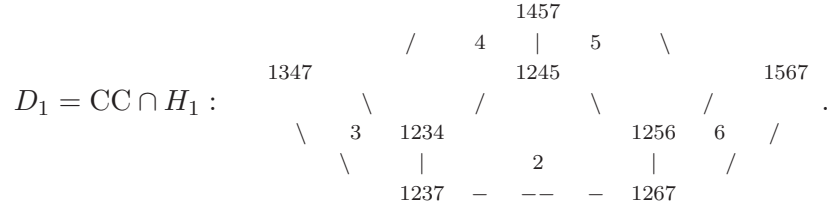
4.3 Seven hyperplanes in the 4-space

Seven hyperplanes in the projective 4-space cut out a *unique* chamber CC stable under the action of \mathbb{Z}_7 . It is bounded by the seven dumplings. We study how the seven dumplings tessellate a 3-sphere.

Figure 41: Blowing up a segment to be a prism (blowup56)

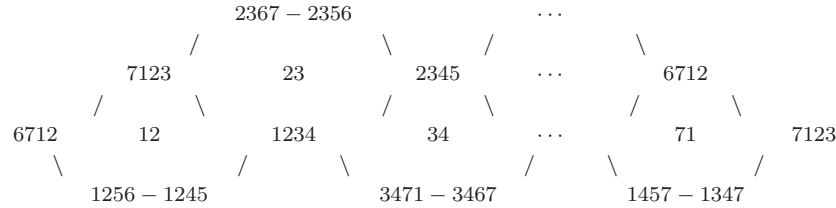
4.3.1 Labeling and a study of 2-skeleton

The central chamber CC has seven faces D_1, \dots, D_7 , which form the \mathbb{Z}_7 -orbit of the dumpling

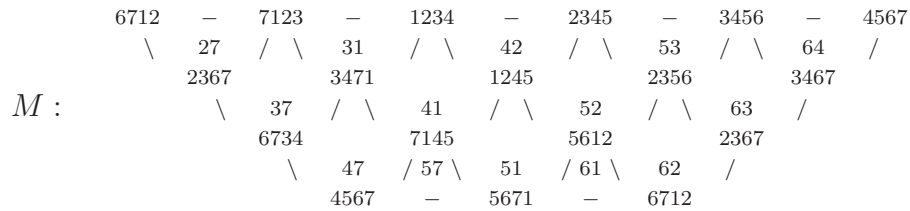


Note that the pentagonal face of D_1 behind is $D_1 \cap D_7$ and the pentagon in front is $D_1 \cap D_2$.² The 2-skeleton of CC consists of seven pentagons, seven triangles and seven rectangles. Remove the intersections of the two consecutive ones:

$$D_1 \cap D_2, \quad D_2 \cap D_3, \quad \dots, \quad D_7 \cap D_1 :$$



(Note that the union of these seven pentagons form a Möbius strip, but for a while just forget it.) Then the remaining seven triangles and seven rectangles form a Möbius strip



Möbius 7

²Using the notation introduced in [CY], CC is represented by $++++++$, and its boundary consists of

$$D_1 = 0+++++, \quad D_2 = -0++++, \quad \dots, \quad D_7 = -----0.$$

The boundary of D_1 consists of

$$\begin{array}{lll} 00++++ : 5\text{-gon} & 0-0++++ : \text{triangle} & 0--0+++ : \text{rectangle} \\ 0---0+++ : \text{rectangle} & 0-----0+ : \text{triangle} & 0-----0 : 5\text{-gon} \end{array}$$

Its boundary

$$C = \partial M : \quad 1234 - 2345 - 3456 - 4567 - 5671 - 6712 - 7123 - 1234$$

is an unknotted circle because C bounds also the disc

$$\begin{array}{ccccccc}
7123 & - & & 1234 & - & & 2345 \\
| & \backslash & 13 & / & \backslash & 24 & / & | \\
| & & 3471 & 14 & & 1245 & & | \\
| & & & \backslash & / & & & | \\
6712 & & 71 & 7145 & & 45 & & 3456 \\
& & \backslash & / & 57 & \backslash & / & \\
& & 5671 & - & & 4567 & &
\end{array}$$

4.3.2 From a solid torus

We start with a vertical pentagonal prism, horizontally sliced into seven thinner pentagonal prisms. Its top and the bottom pentagons are identified after a $4\pi/5$ rotation; this makes the vertical prism a solid torus with a tessellation of the lateral boundary into 35 rectangles. Now collapse vertically seven diagonal rectangles. This changes 21 rectangles (out of 35) into seven segments (forming a circle) and fourteen triangles. Accordingly, the seven thinner pentagonal prisms into dumplings. Next use each collapsed edge as a valley of each pair of triangles face-to-face in order to close the pair like a book, and further identify seven pairs of rectangles. This identification changes the solid torus into a sphere tessellated by seven dumplings.

A vertical pentagonal prism horizontally sliced. We show the above process by using the labels 1, 2, 3, 4, 5, 6, 7 of the seven hyperplanes. The boundary of the vertical pentagonal prism; each line, say 71 is a pentagon on the plane $7 \cap 1$. Two pentagons 71 and 12 sandwich a thin pentagonal prism. 6712 is the point $6 \cap 7 \cap 1 \cap 2$.

71 :	7126	-	7123	-	7134	-	7145	-	7156	-	7126
		×		13		14		15		16	
12 :	1267	-	1237	-	1234	-	1245	-	1256	-	1267
		27		×		24		25		26	
23 :	2367	-	2371	-	3412	-	2345	-	2356	-	2367
		37		31		×		35		36	
34 :	3467	-	3471	-	3412	-	3452	-	3456	-	3467
		47		41		42		×		46	
45 :	4567	-	4571	-	4512	-	4523	-	4563	-	4567
		57		51		52		53		×	
56 :	5674	-	5671	-	5612	-	5623	-	5634	-	5674
		×		61		62		63		64	
67 :	6745	-	6715	-	6712	-	6723	-	6734	-	6745
		75		×		72		73		74	
71 :	7145	-	7156	-	7126	-	7123	-	7134	-	7145
		15		16		×		13		14	

Table of piles 7

Collapsing rectangles. If we identify two points with the same *consecutive* indices as *sets*, for exmaple, $7123 = 1237 = 2371$ but $2356 \neq 5623$, you find seven rectangles

with pairwise coincide vertices (marked by \times), situated diagonally. We collapse such rectangles vertically to a segment, each called a **special edge**. Note that a special edge has two ends which are labeled by consecutive numbers: $\{1234, 2345\}, \{2345, 3451\}, \dots$. Accordingly the remaining rectangles next to the special edges become triangles, and the seven thin rectangular prisms become dumplings. But as a whole it still remains to be a cylinder.

Identifying the top and the bottom to get a solid torus ST . We identify the top pentagon and the bottom pentagon of the collapsed vertical prism according to the labels; note that we must twist the triangle by $4\pi/5$. By the identification we get a solid torus, say ST ; its meridean **mer** is represented by the boundary of a(ny) pentagon, in Table of piles 7, say 71 (or 12, 23, ...). Note that it is still an abstract object. The seven special edges now form a circle:

$$C : 1234 - 2345 - 3456 - 4561 - 5612 - 6712 - 7123 - 1234.$$

We take a vertical line, in Table of piles 7, with a $4\pi/5$ twist, say,

$$\mathbf{par} : 7123 - 3471 - 4571 - 5671 - 7123$$

as a parallel.

The torus T and the solid torus UT . We consider in our space ($\sim \mathbb{S}^3$) a usual (unknotted) solid torus, say UT , and think our solid torus ST fills outside of UT . Let us see the boundary torus

$$T := \partial ST (= \partial UT)$$

has **par** as the meridean, and **mer** as the parallel. From the description of the vertical prism, we see fourteen triangles and fourteen rectangles tessellating the torus T arranged in a hexagonal way (identify the opposite sides of the hexagon):

6712	—	6723	—	6734	—	4567				
	27\72		73		74		75\			
2367	—	7123	—	7134	—	7145	—	5671		
	37		31\13		14		15		16\	
3467	—	3471	—	1234	—	1245	—	1256	—	6712
	47		41		42\24		25		26	
4567	—	4571	—	4512	—	2345	—	2356	—	2367
	\57		51		52		53\35		36	
		5671	—	5612	—	5623	—	3456	—	3467
			\61		62		63		64\46	
				6712	—	6723	—	6734	—	4567

Torus (hexagon) 7

Collapsing the solid torus UT by folding T . We identify two vertices with the same indices as sets. Along each collapsed edge (special edge), there are two triangles with the same vertices; these triangles are folded along the special edge into one triangle. Further there are seven pairs of rectangles with the same vertices; they are also identified. Consequently the torus T is folded along the curve C to be a Möbius strip, which turns out to be the same as the one in the previous subsection (shown as Möbius 7). In this way, we get a space X_7 from the solid torus ST made by six prisms by folding the torus T (collapsing UT).

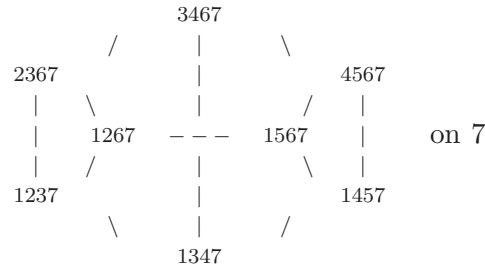
Figure 42: Six dumplings pack a dumpling

Since the curve C is unknotted, this folding is done exactly the same as in the previous section. Then X_7 is homeomorphic to \mathbb{S}^3 .

A model. In [CY], the tessellation is shown, as we quote in Figure 42. In the figure, 0 stands for 7, and the edge connecting 0346 and 0134 is not drawn to avoid complication. Six dumplings are shown; they are packed in a dumpling. The other one is infinitely large. From this diagram, one can hardly see the \mathbb{Z}_7 action. To understand this group action is the motivation of our prism-collapse construction.

4.3.3 $7 - 1 = 6$

If we remove the seventh hyperplane, then the remaining six hyperplanes bounds a hyperhexahedron. This process can be described as follows: the dumpling on the seventh plane reduces to the union of two adjacent segments. More precisely, the two triangles of the boundary of the dumpling reduce to two points, the segment joining the remaining two vertices to a point, and the dumpling to the union of two segments joining the last one to the former ones. Combinatorial explanation: The dumpling on the seventh plane with vertices



reduces to the union of two segments

$$\mathbf{1236} - - \mathbf{1346} - - \mathbf{1456}.$$

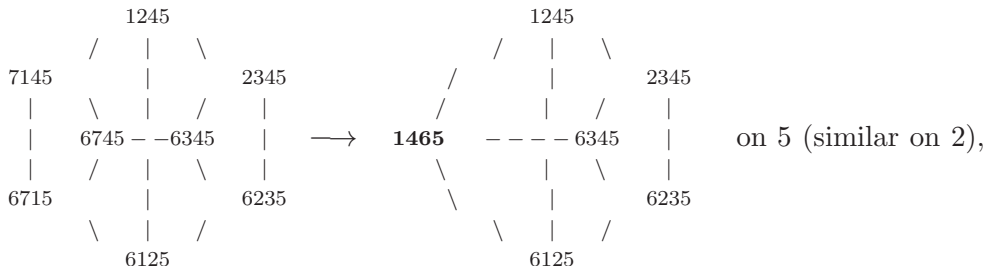
Note that

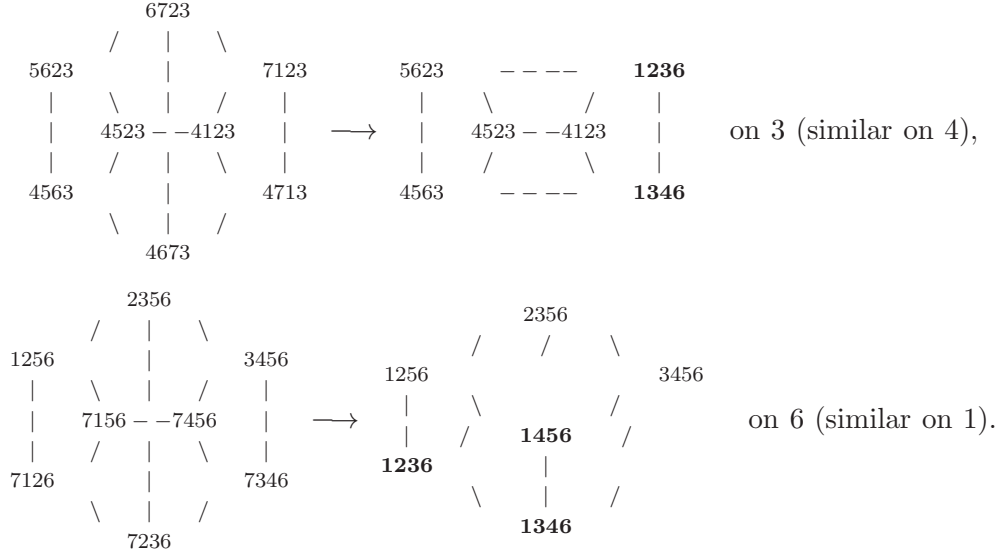
$$\{1, 2, 3, 6\} = \{2, 3, 6, 7\} \cup \{1, 2, 6, 7\} \cup \{1, 2, 3, 7\} - \{7\},$$

$$\{1, 3, 4, 6\} = \{1, 3, 4, 7\} \cup \{3, 4, 6, 7\} - \{7\},$$

$$\{1, 4, 5, 6\} = \{4, 5, 6, 7\} \cup \{1, 5, 6, 7\} \cup \{1, 4, 5, 7\} - \{7\}.$$

Accordingly the other dumplings reduce to prisms (cf. Figure 3), for example,





4.3.4 $6 + 1 = 7$

Let us describe the change of 3-sphere tessellated by six prisms into that by seven dumplings. As we explained in §4.2.4, we express this change – blowing-up a connected union of two edges (of a prism) – by cutting a prism 6 by a hyperplane 7 and dividing the prism into two dumplings, which share the pentagonal face that should be called 6. (See Figure 43).

Figure 43: Blowing up the union of two segments to be a dumpling (blowup67)

4.4 $m(\geq 8)$ hyperplanes in the 4-space

A Veronese arrangement of $m(\geq 8)$ hyperplanes in the projective 4-space cut out a *unique* chamber CC, stable under the action of \mathbb{Z}_m . It is bounded by $m(m-2)$ -dumplings. (An $(m-2)$ -dumpling is bounded by two $(m-2)$ -gons, two triangles and $m-5$ rectangles.) The reader is expected to imagine how these $(m-2)$ -dumplings tessellate a 3-sphere. Here we show a 6-dumpling ($m=8$):

$$D_1 = CC \cap H_1 : \begin{array}{ccccccc} 1458 & - & - & - & - & - & 1568 \\ | & \backslash & & 5 & / & | \\ | & & 1245 & - - & 1256 & & | \\ | & 4 & | & & | & 6 & | \\ 1348 & - & 1234 & 2 & 1267 & - & 1678 \\ & \backslash & 3 & | & | & 7 & / \\ & & 1238 & - - & 1278 & & \end{array} .$$

The central chamber CC has m faces D_1, \dots, D_m , on which \mathbb{Z}_m acts as $D_k \rightarrow D_{k+1} \bmod m$. The 2-skeleton of CC consists of $m(m-2)$ -gons, m triangles and $m(m-5)/2$ rectangles. Remove the intersections of the two consecutive ones:

$$D_1 \cap D_2, \quad D_2 \cap D_3, \quad \dots, \quad D_m \cap D_1.$$

Then the remaining m triangles and $m(m-5)/2$ rectangles form a Möbius strip M , which the reader can make following the previous sections.

Its boundary

$$C = \partial M : \quad 1234 - 2345 - \cdots - m123 - 1234$$

is an unknotted circle because C bounds also a disc. To find a disc is not so obvious so we give a hint when $m = 8$ (the reader is expected to guess the codes for A, B, C, D, E , and then to guess what happens when $m \geq 9$)

$$\begin{array}{ccccccc}
 8123 & - & 1234 & - & 2345 & - & 3456 \\
 | & \backslash & / & \backslash & & / & \backslash & / & | \\
 | & & A & & B & & C & & | \\
 | & & & \backslash & / & & & \backslash & / & | \\
 | & & & D & 15 & & E & & | & \cdot \\
 | & & & & & \backslash & / & & & | \\
 7812 & & 81 & & 8156 & & 56 & & 4567 \\
 & & & \backslash & / & 68 & \backslash & & / \\
 & & 6781 & - & 5678 & & & &
 \end{array}$$

(Answer: $A = 3481, B = 1245, C = 2356, D = 8145, E = 1256$.)

4.4.1 From a solid torus

You start with a vertical $(m-2)$ -gonal prism, horizontally sliced into m thinner $(m-2)$ -gonal prisms. Its top and the bottom triangles are identified after a $4\pi/(m-2)$ rotation. This makes the vertical prism a solid torus; the boundary is tessellated by $m(m-2)$ rectangles.

Now collapse vertically m diagonal rectangles. This changes $3m$ rectangles into m segments (special edges) and $2m$ triangles. Accordingly, the m thinner pentagonal prisms deform into $(m-2)$ -dumplings, forming a solid torus ST . The special edges make a circle C .

We consider in our space ($\sim \mathbb{S}^3$) a usual solid torus, say UT , and think our solid torus ST fills outside of UT .

In the tessellation on the boundary torus

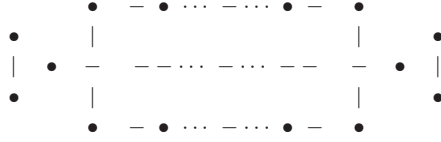
$$T := \partial ST = \partial UT,$$

each special edge is shared by the two triangles with the same code. The special edge is used as a valley to close the pair of triangles like a book, and further identify pairs of the rectangles. This identification changes the torus T into the Möbius strip M with boundary C , and accordingly collapse the solid torus UT , and changes the solid torus ST into a 3-sphere tessellated by $m(m-2)$ -dumplings, recovering the tessellation on ∂CC .

4.4.2 From m hyperplanes to $m-1$ hyperplanes

Consider the central chamber bounded by m hyperplanes. If we remove the m -th hyperplane, then $m-1$ hyperplanes bound the central chamber bounded by $m-1$ hyperplanes. This process can be described as follows: the $(m-2)$ -dumpling on the m -th plane reduces

to $m - 4$ connected segments. Combinatorial explanation: The $(m - 2)$ -dumpling on the m -th plane with vertices



reduces to the union of $m - 4$ segments



Accordingly, the other $(m - 2)$ -dumplings reduce to $(m - 3)$ -dumplings.

4.4.3 From m hyperplanes to $m + 1$ hyperplanes

Let us describe the change of 3-sphere tessellated by m $(m - 2)$ -dumplings into that by $m + 1$ $(m - 1)$ -dumplings. As we explained in §4.2.4, we express this change – blowing-up a connected union of $m - 4$ edges (of a $(m - 2)$ -dumpling) – by cutting the $(m - 2)$ -dumpling (labeled by m) by the hyperplane (labeled by $m + 1$) and dividing the $(m - 2)$ -dumpling into two $(m - 1)$ -dumplings, which share the $(m - 1)$ -gonal face which should be labeled by m . (See Figure 44).

Figure 44: Blowing up the union of $m - 4$ segments to be a $(m - 1)$ -dumpling (blowup89)

5 Appendix: Higher dimensional cases

We describe higher dimensional dumplings D_n for odd n , and central chambers CC_n for even n in arrangements of $n + 3$ hyperplanes in \mathbb{P}^n . Description will be **just combinatorial**. We start from dimension 0: 0-dimensional central chamber CC_0 is a point.

5.1 Review

- **D_1** : 1-dimensional dumpling is a segment, whose boundary consists of two CC_0 's.
- **CC_2** : The 2-dimensional central chamber CC_2 (usually called a pentagon) is bounded by five D_1 's. Each D_1 share CC_0 -faces with two D_1 's.
- **D_3** : Consider the direct product of CC_2 and an interval I . Choose a boundary component of CC_2 , and call it **D_1** . Push down $D_1 \times I$ to D_1 , to get a dumpling D_3 . The crashed **D_1** is called the **special** edge of D_3 . Accordingly, for a boundary component D_1 adjacent to **D_1** (there are two of them), the rectangle $D_1 \times I$ is pushed to become a triangle. For a boundary component D_1 non-adjacent to **D_1** (there are two of them), the rectangle $D_1 \times I$ leaves as it is. Thus the boundary of a dumpling consists of two CC_2 's and two triangles and two quadrilaterals (The following notation is explained in [CY]):

$$\begin{aligned}
 \partial(++++) &= 0++++ \cdots CC_2 \\
 &\cup -0++++ \cdots \text{triangle} \\
 &\cup --0+++ \cdots \text{quadrilateral} \\
 &\cup ---0++ \cdots \text{quadrilateral} \\
 &\cup ----0+ \cdots \text{triangle} \\
 &\cup -----0 \cdots CC_2
 \end{aligned}$$

- **CC_4** : The 4-dimensional central chamber CC_4 is bounded by seven D_3 's. Each D_3 shares CC_2 -faces with two D_3 's. The 2-skeleton of the D_3 -tessellation of \mathbb{S}^3 minus the seven CC_2 's form a Möbius strip M_2 with boundary C_1 consisting of seven special edges D_1 's of seven D_3 's.

On the other hand, we start with a vertical prism with base CC_2 , horizontally sliced into seven thinner prisms with base CC_2 . Collapse the seven diagonal $D_1 \times I$'s on the boundary into seven **D_1** 's; this makes each thinner prism a D_3 with the said **D_1** as special one. Identify the top and the bottom CC_2 after $4\pi/5$ rotation so that the seven special edges **D_1** form a circle C_1 . In this way, we get a solid torus ST_3 made of seven D_3 's.

The (tessellated) boundary torus T_2 of ST_3 is just the double of the (tessellated) Möbius strip M_2 branching along C_1 .

Fold the boundary torus T_2 along the curve C_1 to M_2 , Then the solid torus ST_3 changes into the tessellated boundary of CC_4 .

5.2 D_{2k-1}, D_{2k} ($k \geq 3$)

- **D_{2k-1}** : Consider the direct product of CC_{2k-2} and an interval I . Choose a boundary component D_{2k-3} of CC_{2k-2} , and call it **D_{2k-3}** . Push down $D_{2k-3} \times I$ to **D_{2k-3}** , to get a dumpling D_{2k-1} . The resulting **D_{2k-3}** is called the **special** edge of **D_{2k-1}** .

- **CC_{2k}**: The $2k$ -dimensional central chamber CC_{2k} is bounded by $2k + 3$ dumplings D_{2k-1} . Each D_{2k-1} shares CC_{2k-2} -faces with two D_{2k-1} 's. The $(2k - 2)$ -skeleton of the D_{2k-1} -tessellation of \mathbb{S}^{2k-1} minus the $2k + 3$ chambers CC_{2k-2} form a CW-complex M_{2k-2} with boundary C_{2k-3} ($\sim \Delta_{2k-4} \times \mathbb{S}^1$) consisting of $2k + 3$ special edges \mathbf{D}_{2k-3} of $2k + 3$ dumplings D_{2k-1} .

On the other hand, we start with a vertical prism with base CC_{2k-2} , horizontally sliced into $2k + 3$ thinner prisms with base CC_{2k-2} . Collapse the $2k + 3$ diagonal $D_{2k-3} \times I$ on the boundary into $2k + 3$ dumplings \mathbf{D}_{2k-3} ; this makes each thinner prism a dumpling D_{2k-1} with the said \mathbf{D}_{2k-3} as special edge. Identify the top and the bottom CC_{2k-2} after $4\pi/(2k + 1)$ rotation so that the $2k + 3$ special edges \mathbf{D}_{2k-3} form C_{2k-3} . In this way, we get a solid torus ST_{2k-1} ($\sim \Delta_{2k-2} \times \mathbb{S}^1$) made of $2k + 3$ dumplings D_{2k-1} .

The (tessellated) boundary torus T_{2k-2} ($\sim \mathbb{S}^{2k-3} \times \mathbb{S}^1$) of ST_{2k-1} can be folded along the solid $(2k - 3)$ -torus C_{2k-3} (as a hinge) to M_{2k-2} . Then the solid torus ST_{2k-1} changes into the tessellated boundary of CC_{2k} .

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